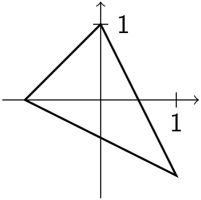
Dylan Thurston diagrams and cluster integrable systems.

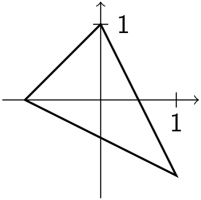
Dylan Thurston diagrams and cluster integrable systems.

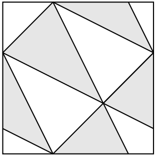
We, Russians, consider examples.
V.I.Arnol'd

The simplest integrable system.

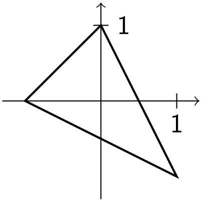


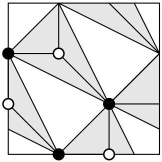
The simplest integrable system.



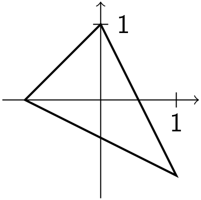


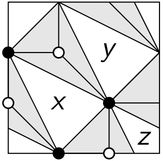
The simplest integrable system.





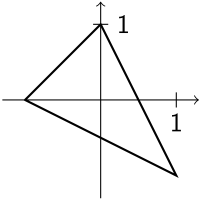
The simplest integrable system.

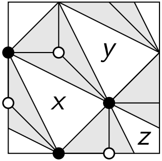




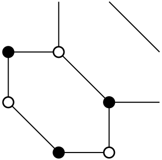
$s_0 s_1 s_0 \Lambda$

The simplest integrable system.

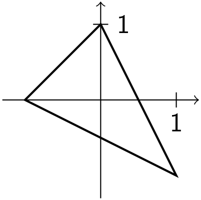


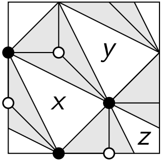


$s_0 s_1 s_0 \Lambda$

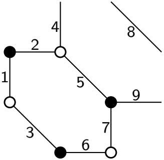


The simplest integrable system.

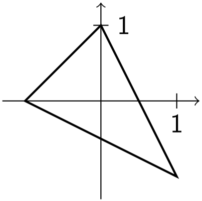


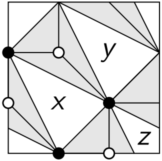


$s_0 s_1 s_0 \Lambda$

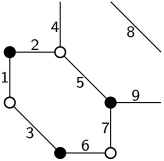


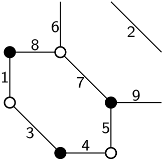
The simplest integrable system.



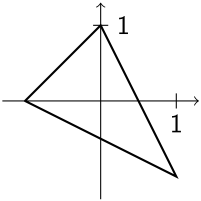


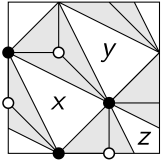
$s_0 s_1 s_0 \Lambda$



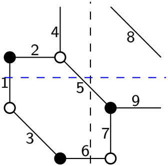


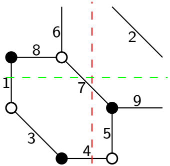
The simplest integrable system.



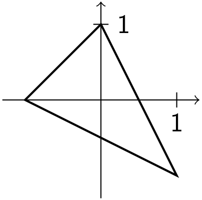


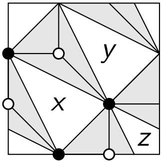
$s_0 s_1 s_0 \Lambda$



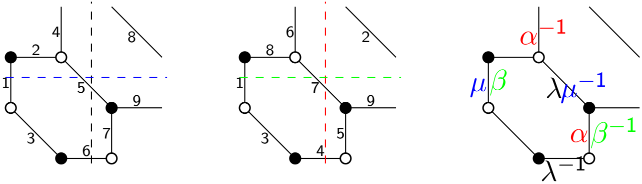


The simplest integrable system.

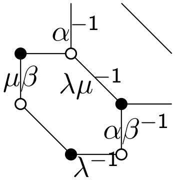




$s_0 s_1 s_0 \Lambda$

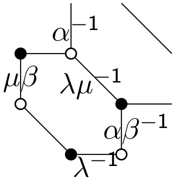


Partition function.



$$x = \alpha\beta^{-2}, \quad y = \alpha\beta, \quad z = \alpha^{-2}\beta$$

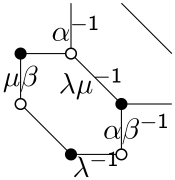
Partition function.



$$x = \alpha\beta^{-2}, \quad y = \alpha\beta, \quad z = \alpha^{-2}\beta$$

$$S^h = \det \begin{pmatrix} \alpha^{-1} & 1 & \lambda\mu^{-1} \\ 1 & \mu\beta & 1 \\ \lambda^{-1} & 1 & \alpha\beta^{-1} \end{pmatrix} =$$

Partition function.

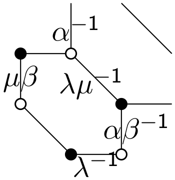


$$x = \alpha\beta^{-2}, \quad y = \alpha\beta, \quad z = \alpha^{-2}\beta$$

$$S^h = \det \begin{pmatrix} \alpha^{-1} & 1 & \lambda\mu^{-1} \\ 1 & \mu\beta & 1 \\ \lambda^{-1} & 1 & \alpha\beta^{-1} \end{pmatrix} =$$

$$= \lambda^{-1} + \mu + \lambda\mu^{-1} - \alpha^{-1} - \beta - \alpha\beta^{-1}.$$

Partition function.



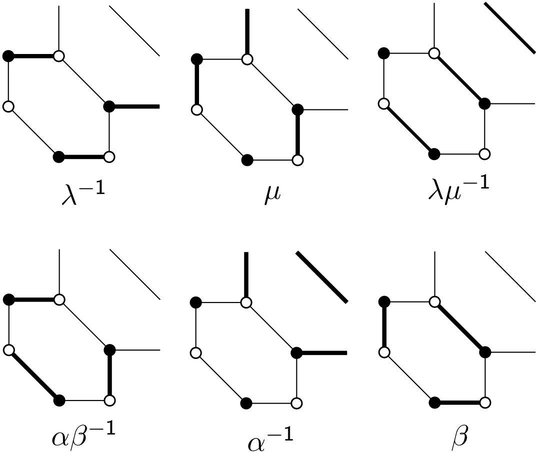
$$x = \alpha\beta^{-2}, \quad y = \alpha\beta, \quad z = \alpha^{-2}\beta$$

$$S^h = \det \begin{pmatrix} \alpha^{-1} & 1 & \lambda\mu^{-1} \\ 1 & \mu\beta & 1 \\ \lambda^{-1} & 1 & \alpha\beta^{-1} \end{pmatrix} =$$

$$= \lambda^{-1} + \mu + \lambda\mu^{-1} - \alpha^{-1} - \beta - \alpha\beta^{-1}.$$

$$S^f = -\lambda^{-1} - \mu - \lambda\mu^{-1} + x^{-1/3}y^{-2/3} + x^{-1/3}y^{1/3} + x^{2/3}y^{1/3}.$$

Partition function as a dimer sum:



$$S^h = \lambda^{-1} + \mu + \lambda\mu^{-1} - \alpha^{-1} - \beta - \alpha\beta^{-1}.$$

Partition function as a characteristic polynomial

$$s_0 s_1 s_0 \Lambda$$

Partition function as a characteristic polynomial

$$s_0 s_1 s_0 \Lambda$$

$$S^f \sim \det\left(T_x \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} T_y \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} T_z \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} - \mu \right) =$$

$$= \mu^2 - \lambda y - \lambda^2 xy - \lambda\mu(1 + y + xy)$$

Partition function as a characteristic polynomial

$$s_0 s_1 s_0 \Lambda$$

$$S^f \sim \det\left(T_x \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} T_y \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} T_z \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} - \mu \right) =$$

$$= \mu^2 - \lambda y - \lambda^2 xy - \lambda\mu(1 + y + xy)$$

Becomes

$$S^f = -\lambda^{-1} - \mu - \lambda\mu^{-1} + x^{-1/3}y^{-2/3} + x^{-1/3}y^{1/3} + x^{2/3}y^{1/3}.$$

After the substitution

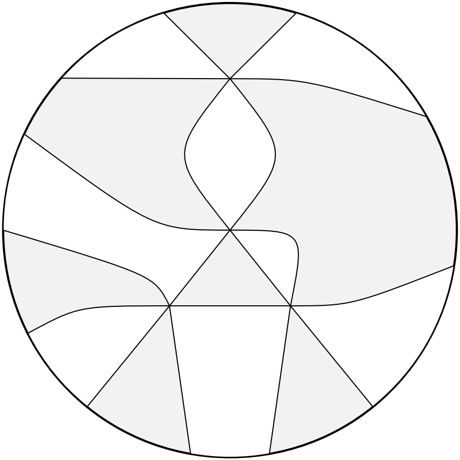
$$\lambda \mapsto \lambda x^{-2/3}y^{-1/3}, \quad \mu \mapsto \mu x^{-1/3}y^{1/3}, \quad S^f \mapsto -S^f x^{2/3}y^{-1/3}$$

Solution.

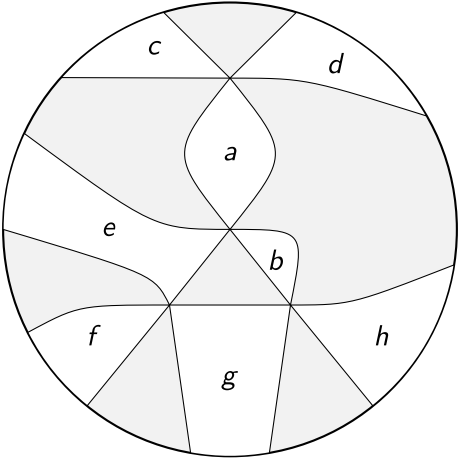
$$x = c_\lambda^{-3}\frac{\theta^3(t-b)}{\theta^3(t)};\quad y = c_\lambda^{3}\frac{\theta^3(t)}{\theta^3(t+b)};\quad z = \frac{\theta^3(t+b)}{\theta^3(t-b)};$$

$$c_\lambda = -\frac{\theta^3(b)}{\theta^3(2b)};\quad 3b \equiv 0$$

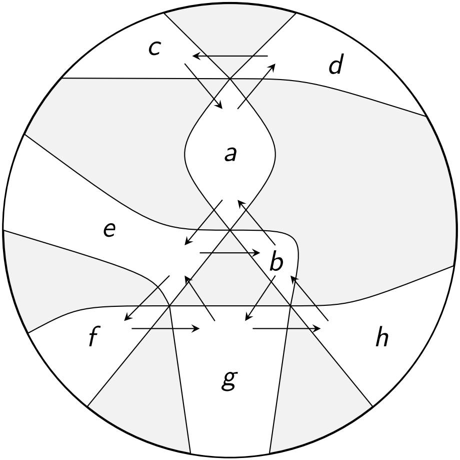
Thurston diagram on a disc.



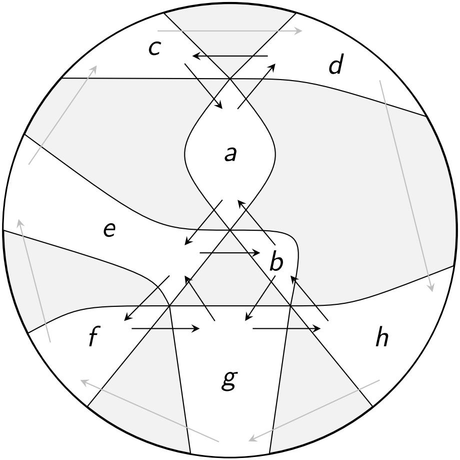
Thurston diagram on a disc.



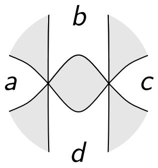
Thurston diagram on a disc.



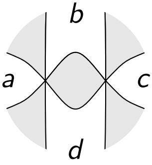
Thurston diagram on a disc.



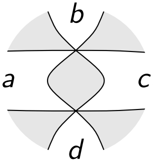
Thurston mutation



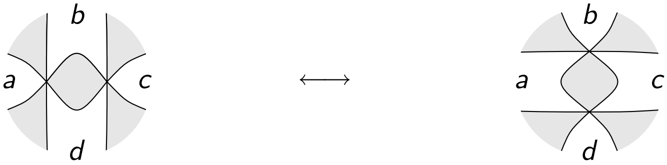
Thurston mutation

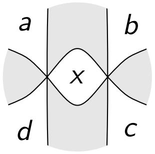


$\longleftrightarrow$

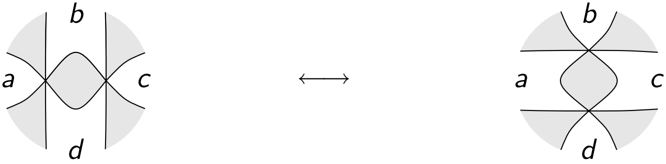


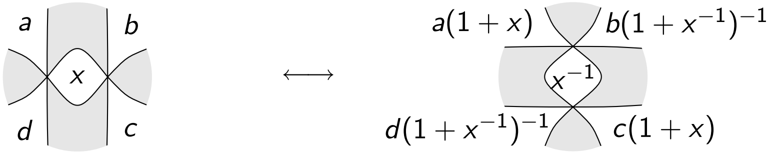
Thurston mutation



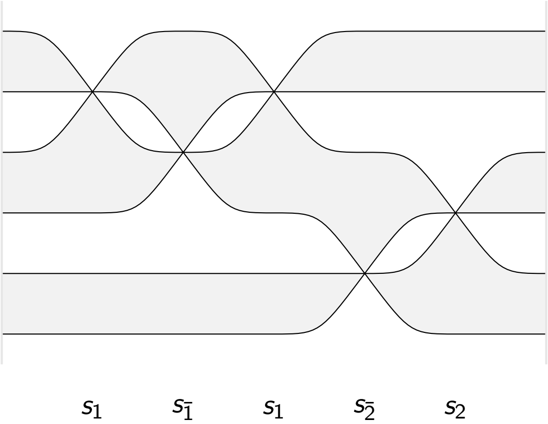


Thurston mutation

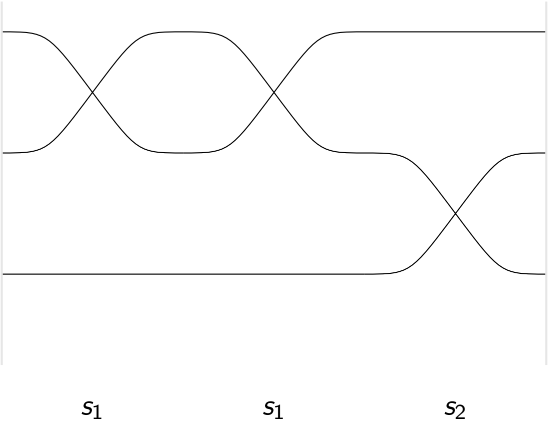




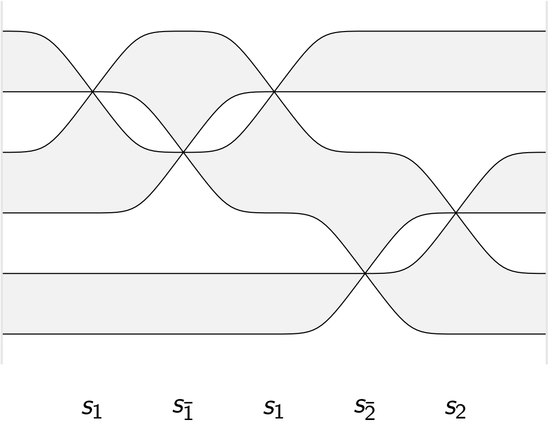
Double permutations



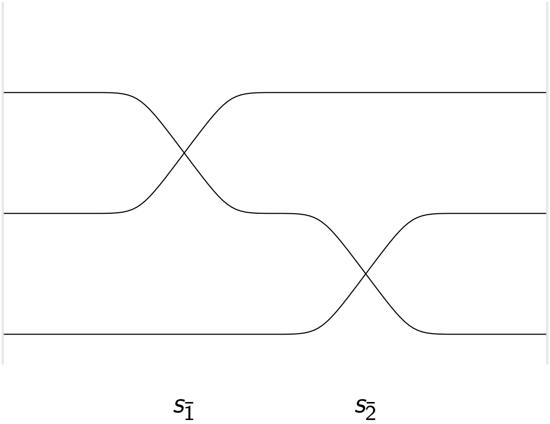
Double permutations



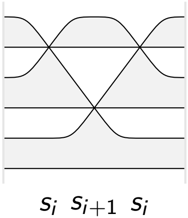
Double permutations



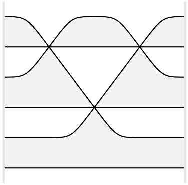
Double permutations



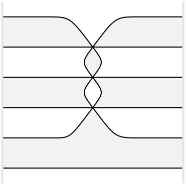
Braid relation



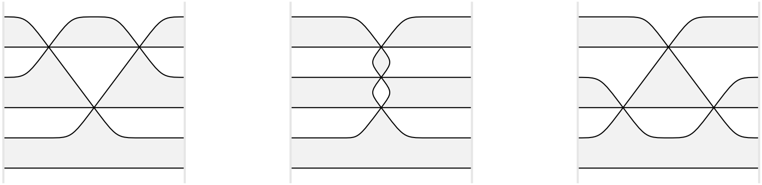
Braid relation



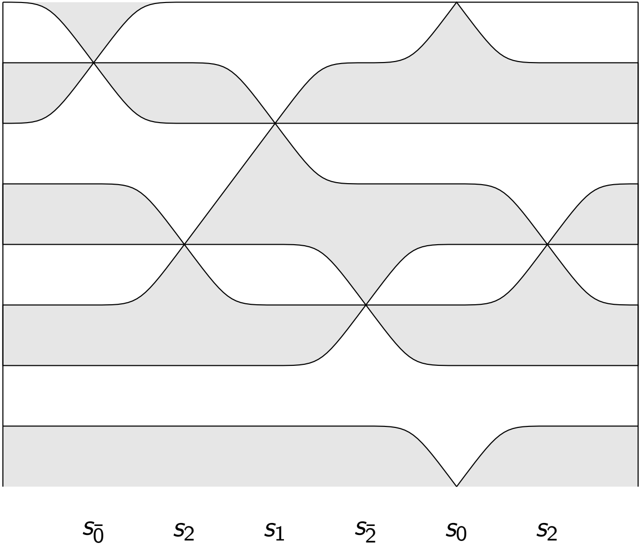
$s_i \; s_{i+1} \; s_i$



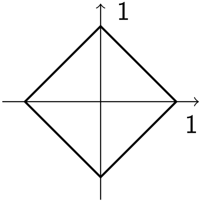
Braid relation

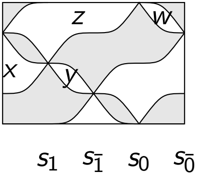


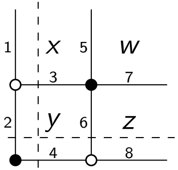
$s_i \; s_{i+1} \; s_i$ $\qquad$ $s_{i+1} \; s_i \; s_{i+1}$

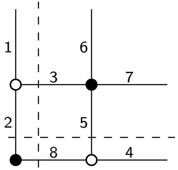


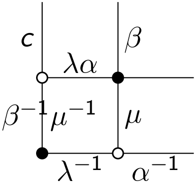
$s_{\bar{0}}$ s_2 s_1 $s_{\bar{2}}$ s_0 s_2


1
1


z
w
x
y
s_1 $s_{\bar{1}}$ s_0 $s_{\bar{0}}$


1
x
5
w
3
7
2
y
6
z
4
8


1
6
3
7
2
5
8
4


c
β
$\lambda\alpha$
β^{-1}
μ^{-1}
μ
λ^{-1}
α^{-1}

$$\{x, y\} = 2xy; \quad \{y, z\} = 2yz; \quad \{z, w\} = 2zw;$$

$$\{w, x\} = 2wx; \quad \{x, z\} = \{y, w\} = 0.$$

$$\{x, y\} = 2xy; \quad \{y, z\} = 2yz; \quad \{z, w\} = 2zw;$$

$$\{w, x\} = 2wx; \quad \{x, z\} = \{y, w\} = 0.$$

$$S^h = \alpha + \alpha^{-1} + c\beta + \beta^{-1} + \lambda + \lambda^{-1} + \mu^{-1} + c\mu$$

$$\{x, y\} = 2xy; \quad \{y, z\} = 2yz; \quad \{z, w\} = 2zw;$$

$$\{w, x\} = 2wx; \quad \{x, z\} = \{y, w\} = 0.$$

$$S^h = \alpha + \alpha^{-1} + c\beta + \beta^{-1} + \lambda + \lambda^{-1} + \mu^{-1} + c\mu$$

$$S^f = x^{-1/2}w^{1/2} + x^{-1/2}w^{-1/2} + x^{1/2}w^{1/2} + x^{1/2}w^{-1/2}wy +$$

$$+\lambda + \lambda^{-1} + \mu^{-1} + yw\mu.$$

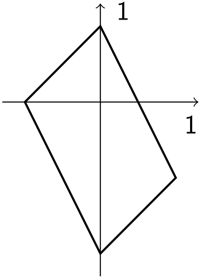
$$\{x, y\} = 2xy; \quad \{y, z\} = 2yz; \quad \{z, w\} = 2zw;$$
$$\{w, x\} = 2wx; \quad \{x, z\} = \{y, w\} = 0.$$

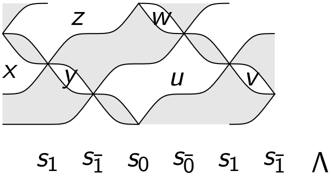
$$S^h = \alpha + \alpha^{-1} + c\beta + \beta^{-1} + \lambda + \lambda^{-1} + \mu^{-1} + c\mu$$

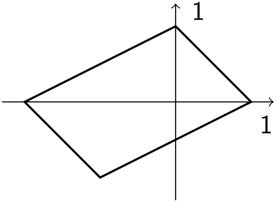
$$S^f = x^{-1/2}w^{1/2} + x^{-1/2}w^{-1/2} + x^{1/2}w^{1/2} + x^{1/2}w^{-1/2}wy +$$
$$+\lambda + \lambda^{-1} + \mu^{-1} + yw\mu.$$

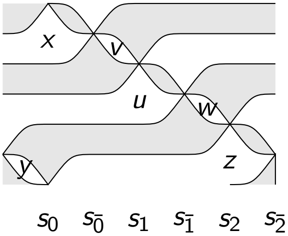
$$(x, y, z, w) \mapsto (z, w, x, y)$$
$$(x, y, z, w) \mapsto (y\frac{(1+x)^2}{(1+z^{-1})^2}, x^{-1}, w\frac{(1+z)^2}{(1+w^{-1})^2}, z^{-1})$$

The second simplest Toda chain.

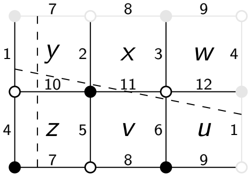




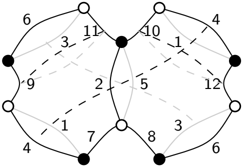




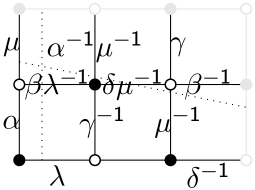
>



7 8 9
1 y 2 x 3 w 4
10 11 12
4 z 5 v 6 u 1
7 8 9



6 11 10 4
3 1
9 2 5 12
1 3
4 7 8 6



μ α−1 μ−1 γ
β λ−1 δ μ−1 β−1
α γ−1 μ−1
λ δ−1

Partition function.

$$S^h = \mu^{-2} + \mu + \mu^{-1}(\alpha\gamma^{-1} + \gamma + \alpha^{-1} + \beta + \alpha\delta + \alpha^{-1}\beta^{-1}\delta^{-1}) +$$

$$+(\gamma\alpha^{-1} + \alpha + \gamma^{-1} + \beta\gamma + \delta + \gamma^{-1}\beta^{-1}\delta^{-1}) + \lambda^{-1} - \lambda\mu^{-1}.$$