Integrable vector evolution equations

Vladimir V. Sokolov
Landau Institute for Theoretical Physics

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Old classification result.

Theorem (Svinolupov-VS 1982). A complete list (up to „almost invertible“ transformations) of equations of the form

$$u_t = u_{xxx} + f(u, u_x, u_{xx}) \tag{1}$$

that have infinite hierarchies of local conservation laws can be written as:

$$\begin{aligned}
u_t &= u_{xxx} + 6u\,u_x, \\
u_t &= u_{xxx} + 6u^2\,u_x, \\
u_t &= u_{xxx} - \frac{1}{2}u_x^3 + (\alpha e^{2u} + \beta e^{-2u})u_x, \\
u_t &= u_{xxx} - \frac{1}{2}Q''\,u_x + \frac{3}{8}\frac{(Q - u_x^2)_x^2}{u_x\,(Q - u_x^2)}, \\
u_t &= u_{xxx} - \frac{3}{2}\,\frac{u_{xx}^2 + Q(u)}{u_x},
\end{aligned}$$

where $Q'''''(u) = 0$.

Vector integrable equations.

Integrable vector evolution equations have the following form:

$$\mathbf{u}_t = f_n\,\mathbf{u}_n + f_{n-1}\,\mathbf{u}_{n-1} + \cdots + f_1\,\mathbf{u}_1 + f_0\,\mathbf{u}, \qquad \mathbf{u}_i = \frac{\partial^i \mathbf{u}}{\partial x^i}. \quad (2)$$

Here $\mathbf{u}$ is N-component vector, the (scalar) coefficients f_i depend on scalar products between $\mathbf{u}, \mathbf{u}_x, \ldots, \mathbf{u}_n$.

We consider equations (2) that are integrable for arbitrary dimension N. In virtue of the arbitrariness of N, all scalar products

$$u_{[i,j]} = (\mathbf{u}_i, \mathbf{u}_j), \qquad i \leqslant j$$

can be regarded as functionally independent variables.

We denote the ring of scalar-valued functions depending on finite number of scalar products by $\mathcal{F}$.

Examples. The following vector mKdV-systems:

$$\mathbf{u}_t = \mathbf{u}_{xxx} + (\mathbf{u}, \mathbf{u})\,\mathbf{u}_x,$$

$$\mathbf{u}_t = \mathbf{u}_{xxx} + (\mathbf{u}, \mathbf{u})\,\mathbf{u}_x + (\mathbf{u}, \mathbf{u}_x)\,\mathbf{u}$$

as well as a vector Harry Dym equation

$$\boldsymbol{u}_t = (\mathbf{u}, \mathbf{u})^{3/2}\boldsymbol{u}_{xxx}$$

are integrable for any N.

It is clear all such equations are invariant with respect to the group $SO(N)$.

Theorem (A.Meshkov, VS 2002).

i). If equation (2) possesses an infinite series of vector commuting flows of the form

$$\mathbf{u}_\tau = g_m\,\mathbf{u}_m + g_{m-1}\,\mathbf{u}_{m-1} + \cdots + g_1\,\mathbf{u}_1 + g_0\,\mathbf{u}, \qquad g_i \in \mathcal{F}, \quad (3)$$

then there exists a formal Lax pair $L_t = [A,\, L]$, where

$$L = a_1\,D_x + a_0\, + a_{-1}\,D_x^{-1} + \cdots\,, \qquad A = \sum_0^n f_i\,D_x^i. \qquad (4)$$

Here f_i are the coefficients of equation (2) and $a_i \in \mathcal{F}$.

ii). The following functions

$$\rho_{-1} = \frac{1}{a_1}, \qquad \rho_0 = \frac{a_0}{a_1}, \qquad \rho_i = \operatorname{res} L^i, \qquad i \in \mathbb{N} \qquad (5)$$

are conserved densities for equation (2).

The conservation laws with densities (5) are called *canonical*.

iii). If equation (2) possesses an infinite series of conserved densities, then there exist the formal Lax operator L and a series S of the form

$$S = s_1 \, D_x + s_0 \, + s_{-1} \, D_x^{-1} + s_{-2} \, D^{-2} + \cdots \, ,$$

such that

$$S_t + A^t \, S + S \, A = 0, \quad S^t = -S, \quad L^t = -S^{-1} L S,$$

where the upper index t stands for a formal conjugation.

iv). Under the conditions of item iii) densities (5) with $i = 2k$ are of the form $\rho_{2k} = D_x(\sigma_k)$ for some functions σ_k.

Idea of the proof.

i). Rewrite equation (2) and its commuting flow (3) in the form

$$\boldsymbol{u}_t = A(\boldsymbol{u}), \qquad \boldsymbol{u}_\tau = B(\boldsymbol{u}), \qquad \text{where} \quad B = \sum_0^m g_i\, D_x^i. \tag{6}$$

Compatibility of (6) leads to the operator identiry

$$B_t - [A,\, B] = A_\tau .$$

For large m we may "ignore"the r.h.s. since it has a small order comparing with the other terms. In other words, the operator B satisfies $L_t = [A,\, L]$ approximately. Then the series of first order $L_m = B^{1/m}$ is an approximate solution as well. The gluing of the first order approximate solutions corresponding to different commuting flows into an exact formal Lax operator L is similar to the scalar case.

ii). It follows from known Adler's theorem.

Hamiltonian and recursion operators.

To define the canonical conserved densities we have considered differential operators and series with scalar coefficients from $\mathcal{F}$. However it is not enough for Hamiltonian and recursion operators and we should extend the ring of coefficients.

Denote by $R_{i,j}$ a $\mathcal{F}$-linear operator that acts on vectors by the rule

$$R_{i,j}(\boldsymbol{v}) = \boldsymbol{u}_i\,(\boldsymbol{u}_j, \boldsymbol{v}).$$

It is easy to see that

$$R_{i,j}R_{p,q} = (\boldsymbol{u}_j, \boldsymbol{u}_p)R_{i,q}, \qquad R_{i,j}^T = R_{j,i}, \qquad \operatorname{trace} R_{i,j} = (\boldsymbol{u}_i, \boldsymbol{u}_j),$$

$$D_x \circ R_{i,j} = R_{i,j}D_x + R_{i+1,j} + R_{i,j+1}.$$

Denote by $\mathcal{O}$ the algebra over $\mathcal{F}$, generated by operators $R_{i,j}$ and by the unity operator.

The Frechet derivatives of elements from $\mathcal{F}$ are differential operators with coefficients from $\mathcal{O}$. For instance, the Frechet derivative of the r.h.s. F of an equation

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2 \mathbf{u}_{xx} + f_1 \mathbf{u}_x + f_0 \mathbf{u} \tag{7}$$

equals

$$F_* = D_x^3 + \sum_k f_k D_x^k + \sum_{i,j,k} \frac{\partial f_k}{\partial u_{[i,j]}} \left(R_{k,i}\, D_x^j + R_{k,j}\, D_x^i \right), \tag{8}$$

where $i, j, k = 0, \ldots, 2$. We will call such differential operators local.

An example of local Hamiltonian operator is given by

$$\mathcal{H} = u_{[0,0]}^2\, D_x + 2\, u_{[0,0]}\, u_{[0,1]} + 2\, u_{[0,0]}\, (R_{0,1} - R_{1,0}).$$

Let us describe all Hamiltonian operators of the form

$$\mathcal{H} = A\, D_x + D_x \circ A + s_4(R_{1,0} - R_{0,1}) +$$

$$s_5(R_{2,0} - R_{0,2}) + s_6(R_{2,1} - R_{1,2}),$$

where

$$A = s_0 + s_1 R_{0,0} + s_2(R_{0,1} + R_{1,0}) + s_3 R_{1,1},$$

and $s_i \in \mathcal{F}$ have second differential degree i.e. depend on $u_{[0,0]}, \ldots, u_{[2,2]}$ and $s_0 \neq 0$.

Proposition. This operator is Hamiltonian iff the coefficients have the following form :

$$s_1 = u_{[0,1]}^2 (s_0\, u_{[0,0]}\, \psi)^{-2} \left(u_{[0,1]} \frac{\partial s_0}{\partial u_{[0,1]}} + 2\, s_0 \right)^2 - \frac{s_0}{u_{[0,0]}},$$

$$s_2 = -u_{[0,1]}^2 (s_0\, \psi)^{-2}\, (u_{[0,0]})^{-1} \frac{\partial s_0}{\partial u_{[0,1]}} \left(u_{[0,1]} \frac{\partial s_0}{\partial u_{[0,1]}} + 2\, s_0 \right),$$

$$s_3 = u_{[0,1]}^2 (s_0\, \psi)^{-2} \left(\frac{\partial s_0}{\partial u_{[0,1]}} \right)^2, \qquad s_5 = -s_2, \qquad s_6 = -s_3,$$

$$s_4 = \frac{u_{[0,1]}^2}{(s_0\, u_{[0,0]}\, \psi)^2} \left(u_{[0,1]} \frac{\partial s_0}{\partial u_{[0,1]}} + 2\, s_0 \right) \times$$

$$\left(u_{[0,1]} \frac{\partial s_0}{\partial u_{[0,1]}} - 4\, s_0 + 2\, u_{[0,0]} \frac{\partial s_0}{\partial u_{[0,0]}} \right) + \frac{6\, u_{[0,1]} \sqrt{s_0}}{\psi\, (u_{[0,0]})^{3/2}} - \frac{s_0}{u_{[0,0]}},$$

where

$$\psi = D_x \left(\frac{(u_{[0,0]})^{1/2}}{s_0^{3/2}} \left(u_{[0,1]} \frac{\partial s_0}{\partial u_{[0,1]}} + 2\, s_0 \right) \right).$$

Here $s_0(u_{[0,0]},\, u_{[0,1]})$ is an arbitrary function. □

If $s_0 = s_0(u_{[0,0]})$, then $s_2 = s_3 = s_5 = s_6 = 0$,

$$s_1 = -\frac{s_0 s_0' (u_{[0,0]} s_0' - 2\, s_0)}{(u_{[0,0]} s_0' - s_0)^2}, \qquad s_4 = -\frac{u_{[0,0]}\, s_0 (s_0')^2}{(u_{[0,0]} s_0' - s_0)^2}.$$

In the case $s_0 = \frac{1}{2}$ we get $\mathcal{H}_1 = D_x$; if $s_0 = \frac{1}{2}\, u_{[0,0]}^2$, we obtain the above example.

Possibly the most Hamiltonian structures for vector integrable equations are non-local. For example, the Hamiltonian operator $\mathcal{H}$ and the symplectic operator $\mathcal{T}$ for the vector MKdV-equation

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{u}, \boldsymbol{u})\, \boldsymbol{u}_x$$

are given by

$$\mathcal{H}(\boldsymbol{w}) = D_x \boldsymbol{w} + (\boldsymbol{u}, D_x^{-1} \circ \boldsymbol{u}) \boldsymbol{w} - (\boldsymbol{u}, D_x^{-1} \circ \boldsymbol{w}) \boldsymbol{u},$$

$$\mathcal{T}(\boldsymbol{w}) = D_x \boldsymbol{w} + \boldsymbol{u} D_x^{-1} \circ (\boldsymbol{u},\, \boldsymbol{w}).$$

Equations of KdV-type.

Integrable vector equations of the form

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2 \mathbf{u}_{xx} + f_1 \mathbf{u}_x + f_0 \mathbf{u} \tag{9}$$

were studied by A.Meshkov and VS.

The coefficients f_i of the equation are scalar functions in the following six variables:

$$(\mathbf{u},\, \mathbf{u}),\, (\mathbf{u}, \mathbf{u}_x),\, (\mathbf{u}_x, \mathbf{u}_x),\, (\mathbf{u}, \mathbf{u}_{xx}),\, (\mathbf{u}_x, \mathbf{u}_{xx}),\, (\mathbf{u}_{xx}, \mathbf{u}_{xx}). \tag{10}$$

Several first canonical densities are given by

$$\rho_0 = -\frac{1}{3}\, f_2, \tag{11}$$

$$\rho_1 = \frac{1}{9}\, f_2^2 - \frac{1}{3}\, f_1 + \frac{1}{3}\, \frac{d}{dx}\, f_2.$$

$$\rho_2 = \frac{1}{3}\, \theta_0 - \frac{1}{3}\, f_0 - \frac{2}{81}\, f_2^3 + \frac{1}{9}\, f_1\, f_2 - \frac{d}{dx}\rho_1 - \frac{1}{3}\, \frac{d^2}{dx^2}\, \rho_0,$$

$$\rho_3 = \frac{1}{3}\, \theta_1 - \frac{d}{dx}\rho_2 - \frac{1}{3}\, \frac{d^2}{dx^2}\rho_1.$$

Formula (11) means that

$$-\frac{1}{3}D_t(f_2) = D_x(\theta_0)$$

for some $\theta \in \mathcal{F}$. Applying the Euler operator

$$\frac{\delta}{\delta \boldsymbol{u}} = \sum_{i \leqslant j} (-D_x)^i \, \boldsymbol{u}_j \Big(\frac{\partial}{\partial u_{[i,j]}}\Big) + (-D_x)^j \, \boldsymbol{u}_i \Big(\frac{\partial}{\partial u_{[i,j]}}\Big) \tag{12}$$

to the both sides of the conservation law, we get

$$0 = \frac{\delta}{\delta \boldsymbol{u}} D_t f_2 = -6\, \boldsymbol{u}_6 \, D_x \Big(\frac{\partial f_2}{\partial u_{[2,2]}}\Big) + \ldots,$$

where the dots mean terms with $\boldsymbol{u}_i, \; i < 6$. Hence

$$f_2 = c\, u_{[2,2]} + \cdots$$

and so on.

If equation has infinitly many conservation laws then ρ_0 is trivial i.e. f_2 is a total x-derivative. It is convenient to rewrite such equation as

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - \frac{3}{2}\,\frac{d\ln f}{dx}\,\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + f_0\boldsymbol{u}, \tag{13}$$

где $\operatorname{ord} f \leqslant 1$.

Proposition. For equation (13) that satisfies the second integrability condition the coefficient f_1 has the following form

$$f_1 = c_1\frac{u_{[2,2]}}{f} + h_1 u_{[1,2]}^2 + h_2 u_{[0,2]}^2 + h_3 u_{[1,2]} u_{[0,2]} + h_4 u_{[1,2]} + h_5 u_{[0,2]} + h_6.$$

Here $\operatorname{ord} h_i \leqslant 1$, and c_1 is a constant. □

It is possible to separate integrable equations into cases:

A. $f = 1$; **B**. $f = f(u_{[0,0]})$; **C**. $f = f(u_{[0,0]}, u_{[0,1]})$;

D. $f = f(u_{[0,0]}, u_{[0,1]}, u_{[1,1]})$.

Shift-invariant equations.

For some special classes the classification is completed. In paticular all equations of the case **A** are found.

Moreover, all equations of the form

$$\boldsymbol{u}_t = D_x \left(\boldsymbol{u}_{xx} + g_1 \boldsymbol{u}_x + g_0 \boldsymbol{u} \right)$$

are listed. We present the classification result in the potencial form

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2 \mathbf{u}_{xx} + f_1 \mathbf{u}_x + f_0 \mathbf{u},$$

where f_i depend on $(\mathbf{u}_x, \mathbf{u}_x), \ (\mathbf{u}_x, \mathbf{u}_{xx}), \ (\mathbf{u}_{xx}, \mathbf{u}_{xx})$ only. It is clear that such equations are invariant w.r.t. translations $\boldsymbol{u} \to \boldsymbol{u} + \boldsymbol{c}$.

List 1:

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + \frac{3}{2}\left(\frac{a^2\, {u_{[1,2]}}^2}{1 + a\, u_{[1,1]}} - a\, u_{[2,2]}\right)\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{{u_{[1,2]}}^2}{{u_{[1,1]}}^2(1 + a\, u_{[1,1]})}\right)\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - \frac{3}{2}\,(p+1)\,\frac{u_{[1,2]}}{p\, u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\,(p+1)\left(\frac{u_{[2,2]}}{u_{[1,1]}} - \frac{a\, {u_{[1,2]}}^2}{p^2\, u_{[1,1]}}\right)\boldsymbol{u}_x,$$

Here a is a constant and $p = \sqrt{1 + a\, u_{[1,1]}}$. Notice that if $a = 0$ the latter equation is reduced to

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + 3\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_x,$$

Auto-Bäcklund transformations.

The auto-Bäcklund transformations of the first order is defined by

$$\boldsymbol{u}_x = h\,\boldsymbol{v}_x + f\,\boldsymbol{u} + g\,\boldsymbol{v}, \tag{14}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions of the same vector equation. The functions f, g and h are (scalar) functions in variables

$$u_{[0,0]}{=}(\boldsymbol{u},\boldsymbol{u}), \qquad v_{[i,j]} \stackrel{def}{=} (\boldsymbol{v}_i, \boldsymbol{v}_j), \qquad w_i \stackrel{def}{=} (\boldsymbol{u},\, \boldsymbol{v}_i), \qquad i,j \geqslant 0.$$

Example. The Bäcklund transformation for the vector Swartz-KdV equation

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_x$$

is given by

$$\boldsymbol{u}_x = \frac{2\,\mu}{\boldsymbol{v}_x^2}\,(\boldsymbol{u}-\boldsymbol{v},\,\boldsymbol{v}_x)\,(\boldsymbol{u}-\boldsymbol{v}) - \frac{\mu}{\boldsymbol{v}_x^2}\,(\boldsymbol{u}-\boldsymbol{v})^2\,\boldsymbol{v}_x,$$

where μ is an arbitrary parameter.

The superposition formula

$$\boldsymbol{z} = \boldsymbol{u} + (\mu - \nu)\,\frac{\nu\,(\boldsymbol{u}-\boldsymbol{v}')^2\,(\boldsymbol{u}-\boldsymbol{v}) - \mu\,(\boldsymbol{u}-\boldsymbol{v})^2\,(\boldsymbol{u}-\boldsymbol{v}')}{\big(\mu\,(\boldsymbol{u}-\boldsymbol{v}) - \nu\,(\boldsymbol{u}-\boldsymbol{v}')\big)^2},$$

corresponding to this auto-Bäcklund transformation connects 4 different solutions

$$\begin{array}{ccc} \boldsymbol{v}' & \xrightarrow{\ \mu\ } & \boldsymbol{z} \\ {\scriptstyle\nu}\big\uparrow & & \big\uparrow{\scriptstyle\nu} \\ \boldsymbol{u} & \xrightarrow[\ \mu\ ]{} & \boldsymbol{v} \end{array}$$

of the same equation. It defines a known integrable vector discrete model.

Equations on the sphere.

The condition $\boldsymbol{u}^2 = 1$ reduces the number of independent scalar product. Indeed, diffentiating $\boldsymbol{u}^2 = 1$, we obtain

$$(\boldsymbol{u},\, \boldsymbol{u}_x) = 0, \qquad (\boldsymbol{u},\, \boldsymbol{u}_{xx}) = -(\boldsymbol{u}_x,\, \boldsymbol{u}_x)$$

and so on. Moreover, the relation $(\boldsymbol{u},\, \boldsymbol{u}_t) = 0$ specifies f_0. So we consider equations

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + ((\boldsymbol{u}_x,\, \boldsymbol{u}_x)f_2 + 3(\boldsymbol{u}_x,\, \boldsymbol{u}_{xx}))\boldsymbol{u}, \quad (15)$$

where $\quad f_i = f_i((\boldsymbol{u}_x,\, \boldsymbol{u}_x), (\boldsymbol{u}_x,\, \boldsymbol{u}_{xx}),\, (\boldsymbol{u}_{xx},\, \boldsymbol{u}_{xx}))$.

List 2:

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2\,(1 + a\,u_{[1,1]})}\right)\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + \frac{3}{2}\left(\frac{a^2\,u_{[1,2]}^2}{1 + a\,u_{[1,1]}} - a\,(u_{[2,2]} - u_{[1,1]}^2) + u_{[1,1]}\right)\boldsymbol{u}_x + 3\,u_{[1,2]}\,\boldsymbol{u},$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{(p+1)\;u_{[1,2]}}{2\,p\,u_{[1,1]}}\,\boldsymbol{u}_{xx} + 3\,\frac{(p-1)\;u_{[1,2]}}{2\,p}\,\boldsymbol{u}$$

$$+\frac{3}{2}\left(\frac{(p+1)\,u_{[2,2]}}{u_{[1,1]}} - \frac{(p+1)\,a\,{u_{[1,2]}}^2}{p^2 u_{[1,1]}} + u_{[1,1]}\,(1-p)\right)\boldsymbol{u}_x.$$

Anisotropic equations.

Example. Consider equation (I.Golubchik-VS 2000):

$$\boldsymbol{u}_t = \left(\boldsymbol{u}_{xx} + \frac{3}{2}(\boldsymbol{u}_x,\ \boldsymbol{u}_x)\boldsymbol{u}\right)_x + \frac{3}{2}\langle \boldsymbol{u}, \boldsymbol{u}\rangle\, \boldsymbol{u}_x, \qquad \boldsymbol{u}^2 = 1. \quad (16)$$

Here $\langle \boldsymbol{a},\, \boldsymbol{b}\rangle = (\boldsymbol{a},\, R\,\boldsymbol{b})$, where $R = \mathrm{diag}(r_1, ..., r_N)$ is arbitrary constant matrix.

Equation (16) has a Lax representation $L_t = [A, L],$, where

$$L = D_x + \begin{pmatrix} 0 & \Lambda \mathbf{u} \\ \mathbf{u}^T \Lambda, & 0 \end{pmatrix}.$$

Here

$$\Lambda = \frac{1}{\lambda} \mathrm{diag}\,(\sqrt{1 - \lambda^2 r_1}, \ldots, \sqrt{1 - \lambda^2 r_N}).$$

It was a first explicit example of a Lax operator with the spectral parameter lying on the algebraic curve

$$\lambda_1^2 + r_1 = \lambda_2^2 + r_2 - \cdots = \lambda_N^2 + r_N$$

of genus $1 + (N-3)2^{N-2}$.

If $N = 3$, then (16) is a commuting flow of the Landau - Lifshitz equation.

Becides, (16) defines a commuting flow for the Noemann system

$$\boldsymbol{u}_{xx} = -\Big((\boldsymbol{u}_x,\, \boldsymbol{u}_x) + (\boldsymbol{u},\, R\,\boldsymbol{u})\Big)\,\boldsymbol{u} + R\,\boldsymbol{u}, \qquad \boldsymbol{u}^2 = 1,$$

describing the dynamics of a particle on the sphere under the quadratic potential $\mathcal{U} = \frac{1}{2}(\boldsymbol{u},\, R\,\boldsymbol{u})$. More precisely, if we eliminate the derivatives $\boldsymbol{u}_{xx}$ and $\boldsymbol{u}_{xxx}$ from (16), then the reduced system

$$\boldsymbol{u}_t = \frac{1}{2}\Big((\boldsymbol{u}_x,\, \boldsymbol{u}_x) + (\boldsymbol{u},\, R\,\boldsymbol{u})\Big)\,\boldsymbol{u}_x - (\boldsymbol{u}_x,\, R\,\boldsymbol{u})\,\boldsymbol{u} + R\,\boldsymbol{u}_x$$

is a commuting flow for the Noemann system. $\square$

In this example the coefficients of vector equation (2) depend on two different independent scalar products $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$. We call such equations *anisotropic*.

All anisotropic equations

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2\mathbf{u}_{xx} + f_1\mathbf{u}_x + f_0\mathbf{u},$$

on the sphere $u_{[0,0]} = 1$ have been found. In this case the coefficients f_i depend on

$$u_{[1,1]},\, u_{[1,2]},\, u_{[2,2]}, \quad v_{[0,0]},\, v_{[0,1]},\, v_{[1,1]},\, v_{[0,2]},\, v_{[1,2]},\, v_{[2,2]},$$

where

$$v_{[i,j]} = (\boldsymbol{u}_i,\, R(\boldsymbol{u}_j)).$$

List 3:

All „rational“ equations of the list are:

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \left(\frac{3}{2}\, u_{[1,1]} + v_{[0,0]}\right) \boldsymbol{u}_1 + 3\, u_{[1,2]}\, \boldsymbol{u}_0,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\, \frac{u_{[1,2]}}{u_{[1,1]}}\, \boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} + \frac{v_{[1,1]}}{u_{[1,1]}}\right) \boldsymbol{u}_1,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} - \frac{(v_{[0,1]} + u_{[1,2]})^2}{q\, u_{[1,1]}} + \frac{v_{[1,1]}}{u_{[1,1]}}\right)\boldsymbol{u}_1,$$

where $q = u_{[1,1]} + v_{[0,0]} + a.$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{v_{[0,1]}}{v_{[0,0]}}\boldsymbol{u}_2 - 3\left(\frac{v_{[0,2]}}{v_{[0,0]}} - 2\frac{v_{[0,1]}^2}{v_{[0,0]}^2}\right)\boldsymbol{u}_1 + 3\left(u_{[1,2]} - \frac{v_{[0,1]}}{v_{[0,0]}}u_{[1,1]}\right)\boldsymbol{u},$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{v_{[0,1]}}{v_{[0,0]}}\boldsymbol{u}_2 - 3\left(\frac{2v_{[0,2]} + v_{[1,1]} + a}{2v_{[0,0]}} - \frac{5}{2}\frac{v_{[0,1]}^2}{v_{[0,0]}^2}\right)\boldsymbol{u}_1 +$$

$$+3\left(u_{[1,2]} - \frac{v_{[0,1]}}{v_{[0,0]}}u_{[1,1]}\right)\boldsymbol{u},$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{v_{[0,1]}}{v_{[0,0]}}\left(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}\right) + 3u_{[1,2]}\boldsymbol{u} +$$

$$+\frac{3}{2}\Bigg(- \frac{u_{[2,2]}}{v_{[0,0]}} + \frac{(u_{[1,2]} + v_{[0,1]})^2}{v_{[0,0]}(v_{[0,0]} + u_{[1,1]})} +$$

$$+\frac{(v_{[0,0]} + u_{[1,1]})^2}{v_{[0,0]}} + \frac{v_{[0,1]}^2 - v_{[0,0]}\, v_{[1,1]}}{v_{[0,0]}^2}\Bigg)\boldsymbol{u}_1,$$

A hyperbolic integrable equation on the sphere is given by

$$\boldsymbol{u}_{xy} = \frac{\boldsymbol{u}_x}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle} \Big(\langle \boldsymbol{u}, \boldsymbol{u}_y \rangle + \sqrt{1 + \langle \boldsymbol{u}, \boldsymbol{u} \rangle (\boldsymbol{u}_x, \boldsymbol{u}_x)^{-2}}\, \varphi \Big) - (\boldsymbol{u}_x, \boldsymbol{u}_y) \boldsymbol{u},$$

$$\text{where} \qquad \varphi = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u}_y \rangle^2 + \langle \boldsymbol{u}, \boldsymbol{u} \rangle (1 - \langle \boldsymbol{u}_y, \boldsymbol{u}_y \rangle)}.$$

In the case $N = 2$ this equation is equivalent to

$$u_{xy} = sn(u) \sqrt{u_x^2 + 1} \sqrt{u_y^2 + 1}.$$