Baxter operators, Hecke algebras and L-factors

Dmitry Lebedev

ITEP, Moscow and MPIM, Bonn

Dedicated to 65 birthday of Misha Semenov-Tyan-Shansky

References

- My talk is based on common papers with A.Gerasimov [GL] and A.Gerasimov and S.Oblezin [GLO]: [GLO] , "Baxter operator formalism for Macdonald polynomials" arXiv: 1204.0926v2.

GL " On universal Baxter operator for classical groups " arXiv: 1104.0420v1.

GLO " On q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions I-III" arXiv: 0803.0145; 0803.0970; 0805.3754.

GLO "Baxter operator and archimedean Hecke algebras" arXiv: 0706.3476.

GLO " On a classical limit of q-deformed Whittaker function" arXiv: 1101.4567

Plan

- Baxter operator formalism for Macdonald polynomials (q, t).
- Limit $t \longrightarrow \infty$ to q-Whittaker functions.
- Limit $q \longrightarrow 1$–the case of Toda chain.
- Baxter operator $=$ Hecke.
- Connection with classical theory of automorphic L-functions (Piatetsky-Shapiro and Rallis work).
- Conclusion

Macdonald symmetric polynomials [Macdonald]

- Let $\mathbb{Q}(q,t)$ be a field of rational functions in variables q and t. Let $\Lambda_{q,t}$ be the graded $\mathbb{Q}(q,t)$-algebra of symmetric polynomials of variables $x_1, x_2, \ldots$ of degree one

$$\Lambda_{q,t} = \bigoplus_{n\geq 0} \Lambda_{q,t}^{(n)},$$

  where $\Lambda_{q,t}^{(n)}$ is the homogeneous component of $\Lambda_{q,t}$ of degree n.
- There are various convenient bases in the space of symmetric polynomials in variables $x_1, \ldots, x_{\ell+1}$ enumerated by partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{\ell+1})$, $\lambda_i \in \mathbb{Z}_+$.

- Particularly, the elements of the bases of monomial symmetric functions $m_\lambda(x)$ are given by sums of all distinct monomials obtained from $x^\lambda = x_1^{\lambda_1}\dots x_{\ell+1}^{\lambda_{\ell+1}}$ by permutations of $x_1,\dots,x_{\ell+1}$.
- Let us denote $p_n(x) := m_{(n)}$ the symmetric polynomial for the partition $(n)=(n,0,\dots,0)$. The bases of power series symmetric polynomials consists of the polynomials $p_\lambda(x) = p_{\lambda_1}(x)\cdot\ldots\cdot p_{\lambda_{\ell+1}}(x)$.

- Equip the space $\Lambda_{q,t}^{(\ell+1)}$ with a scalar product $< , >$ defined by

$$< p_\lambda,\, p_\mu >_{q,\,t} = \delta_{\lambda\mu}\, z_\lambda \prod_{i=1}^{\ell+1} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\,,$$

$$z_\lambda = \prod_{n\geq 1} n^{m_n} m_n!\,, \qquad m_n = \big|\{k : \lambda_k = n\}\big|\,.$$

- Macdonald introduced a bases $\{P_\lambda(x) = P_\lambda(x;\, q, t)\}$ of symmetric polynomials over $\mathbb{Q}(q, t)$ enumerated by partitions λ such that

$$P_\lambda(x) \;=\; \sum_{\mu\leq\lambda} u_{\lambda\mu}\, m_\mu\,, \qquad\qquad u_{\lambda\lambda} = 1\,,$$

and

$$< P_\lambda,\, P_\mu >_{q,\,t} = \; 0, \qquad\qquad \lambda \neq \mu\,.$$

In the above formula $\leq$ denotes the natural ordering:

$$\lambda \leq \mu \quad \Longleftrightarrow \quad \lambda_1 + \ldots + \lambda_i \;\leq\; \mu_1 + \ldots + \mu_i\,, \quad i \geq 0\,.$$

- The Macdonald polynomials $P_\lambda(x)$ can be also characterized as common eigenfunctions of the following set of mutually commuting difference operators acting in $\Lambda_{q,t}^{(\ell+1)}$:

$$M_r \,=\, t^{r(r-1)/2} \sum_{I_r} \prod_{\substack{i\in I_r\\ j\notin I_r}} \frac{tx_i - x_j}{x_i - x_j}\, T_{I_r}\,, \qquad\qquad T_{I_r} \,=\, \prod_{i\in I_r} T_{q,\,x_i}\,,$$

where the sum goes over all r-element subsets I_r of $(1, 2, \dots, \ell+1)$ and

$$T_{q,\,x_i} \cdot f(x_1, \dots, x_{\ell+1}) \,=\, f(x_1, \dots, qx_i, \dots, x_{\ell+1}).$$

Baxter operator formalism for Macdonald symmetric polynomials [GLO 2012]

Definition

Baxter operator $\mathcal{Q}_\gamma = \mathcal{Q}_\gamma(q,t)$ associated with Macdonald integrable system is a family of operators acting on the space $\Lambda_{q,t}^{(\ell+1)}$ of symmetric polynomials as follows:

$$\mathcal{Q}_\gamma \cdot P(x) \,=\, \int_T d^\times y\;Q_\gamma(x,y)\,\Delta(y)\,P(y^{-1})\,,\qquad \gamma\in\mathbb{Z}\,,$$

($T=\{...|y_i|=1,...)$) where integral kernel is given by

$$Q_\gamma(x,y) \,=\, \prod_{i=1}^{\ell+1}(x_iy_i)^\gamma\prod_{i,j=1}^{\ell+1}\Gamma_{q,t}(x_iy_j)\,.$$

Where

$$\Gamma_{q,t}(x) \;=\; \frac{(tx;\,q)_\infty}{(x;\,q)_\infty}\,, \qquad (x;\,q)_\infty \;=\; \prod_{j=0}^{\infty}\big(1 - xq^j\big),$$

$$\Delta(y;\,q,t) \;=\; \prod_{\substack{i,j=1\\ i\neq j}}^{\ell+1} \frac{1}{\Gamma_{q,t}(y_i y_j^{-1})}.$$

Theorem

The operator $\mathcal{Q}_\gamma$ acts on the Macdonald polynomials $P_\lambda(x)$ as follows:

$$\mathcal{Q}_\gamma \cdot P_\lambda(x) \,=\, L_\gamma(\lambda)\, P_\lambda(x)\,, \qquad \lambda_{\ell+1} \geq \gamma,\;\; \gamma \in \mathbb{Z},$$

$$\mathcal{Q}_\gamma \cdot P_\lambda(x) \,= 0\,, \qquad \lambda_{\ell+1} < \gamma,$$

where

$$L_\gamma(\lambda) \,:=\, L_\gamma(\lambda, q, t) = \prod_{i=1}^{\ell+1} \frac{\Gamma_{q,\,tq^{-1}}(q)}{\Gamma_{q,\,tq^{-1}}\big(t^{\ell+1-i}q^{\lambda_i-\gamma+1}\big)}\,.$$

Proof.

Based on Cauchy-Littlewood identity [Macdonald]. Consider two sets $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ of variables. Then

$$\prod_{i=1}^{n}\prod_{j=1}^{m}\Gamma_{q,t}(x_i y_j) = \sum_{\lambda\in Y_{n,m}} b_\lambda \, P_\lambda(x) \, P_\lambda(y),$$

where summation goes over a set $Y_{m,n}$ of the partitions of length $\min(m,n)$, $b_\lambda =< P_\lambda, P_\lambda >_{q,t}^{-1}$.

Definition

The dual Baxter operator $\check{\mathcal{Q}}_z = \check{\mathcal{Q}}_z(q, t)$ is a family of operators acting in $\Lambda_{q,t}^{(\ell+1)}$

$$\check{\mathcal{Q}}_z \cdot P_\lambda(x) \,=\, \sum_{\mu \in \mathbb{Z}^{\ell+1}} \check{Q}_z(\lambda,\, \mu)\, P_\mu(x)$$

with the kernel function

$$\check{Q}_z(\lambda,\, \mu) \,=\, z^{|\mu|-|\lambda|}\, \varphi_{\mu/\lambda}\ ,$$

Here

$$\varphi_{\mu/\lambda} = \prod_{\substack{i,j=1 \\ i\le j}}^{\ell+1} \frac{\Gamma_{q,tq^{-1}}(t^{j-i}q^{\mu_i-\mu_j+1})}{\Gamma_{q,tq^{-1}}(t^{j-i}q^{\mu_i-\lambda_j+1})} \frac{\Gamma_{q,tq^{-1}}(t^{j-i}q^{\lambda_i-\lambda_{j+1}+1})}{\Gamma_{q,tq^{-1}}(t^{j-i}q^{\lambda_i-\mu_{j+1}+1})}$$

$$\times\Theta(\mu_1-\lambda_1)\prod_{i=1}^{\ell}\Theta(\lambda_i-\mu_{i+1})\Theta(\mu_{i+1}-\lambda_{i+1}) \ ,$$

where in the product one should omit the factors depending on $\lambda_{\ell+2}$ and $\mu_{\ell+2}$.

Theorem

The action of the dual Baxter operator on the Macdonald polynomials reads

$$\check{\mathcal{Q}}_z \cdot P_\lambda(x) \,=\, L_z^\vee(x)\, P_\lambda(x)\,,$$

where the eigenvalue is given by

$$L_z^\vee(x) \,=\, \prod_{i=1}^{\ell+1} \Gamma_{q,t}(zx_i)\,.$$

Proof.

The proof is based on the following Pieri rules [Macdonald]. Let $P_{(n)}(x)$ be the Macdonald polynomial corresponding to the partition $(n) = (n, 0, \ldots, 0)$. Then the following product decomposition holds:

$$P_{(n)}(x) \times P_\lambda(x) \;=\; b_{(n)}^{-1} \sum_{\substack{\mu_i \geq \lambda_i \geq \mu_{i+1} \\ |\mu| - |\lambda| = n}} \varphi_{\mu/\lambda}\, P_\mu(x)\,,$$

$$b_{(n)} = \prod_{i=1}^{n} \frac{1 - tq^{n-i}}{1 - q^{n+1-i}}.$$

Recursive operators

- Let us introduce the following notation

$$\underline{a}_n \;=\; (a_{n,1},\ldots,a_{n,\,n})\,, \qquad \underline{a}'_n \;=\; (a_{n,1},\ldots,a_{n,\,n-1}).$$

- The following recursive relations hold ([Awata, Odake, Shiraishi]):

$$P_\lambda(\underline{x}_{\ell+1}) \;=\; \int_T d^\times \underline{x}_\ell\; Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1};\, \underline{x}_\ell \,|\, \lambda_{\ell+1})\,\Delta(\underline{x}_\ell)\,P_{\lambda'}(\underline{x}_\ell^{-1})\,,$$

$$Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1};\, \underline{x}_\ell \,|\, \lambda_{\ell+1}) \;=\; x_{\ell+1,\,\ell+1}^{\lambda_{\ell+1}} \prod_{i=1}^{\ell} \big(x_{\ell+1,\,i}\, x_{\ell,\,i}\big)^{\lambda_{\ell+1}} \times$$

$$\prod_{i=1}^{\ell+1}\prod_{j=1}^{\ell} \Gamma_{q,t}(x_{\ell+1,\,i},\, x_{\ell,\,j})\,,$$

- The dual recursive relation ([Macdonald]):

$$P_{\underline{\lambda}_{\ell+1}}(x) \;=\; \sum_{\underline{\lambda}_\ell} \overset{\vee}{Q}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{\lambda}_{\ell+1},\,\underline{\lambda}_\ell\,|\,x_{\ell+1})\,P_{\underline{\lambda}_\ell}(x')\,,$$

$$\overset{\vee}{Q}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{\lambda}_{\ell+1},\,\underline{\lambda}_\ell\,|\,x_{\ell+1}) \;=\; x_{\ell+1}^{|\underline{\lambda}_{\ell+1}|-|\underline{\lambda}_\ell|}\psi_{\underline{\lambda}_{\ell+1}/\underline{\lambda}_\ell}\,,$$

- $$\psi_{\lambda/\mu} = \prod_{1\leq i\leq j\leq \ell} \frac{\Gamma_{q,\,tq^{-1}}\big(t^{j-i}q^{\mu_i-\mu_j+1}\big)}{\Gamma_{q,\,tq^{-1}}\big(t^{j-i}q^{\lambda_i-\mu_j+1}\big)}\,\frac{\Gamma_{q,\,tq^{-1}}\big(t^{j-i}q^{\lambda_i-\lambda_{j+1}+1}\big)}{\Gamma_{q,\,tq^{-1}}\big(t^{j-i}q^{\mu_i-\lambda_{j+1}+1}\big)}$$

  when $\lambda_1 \geq \mu_1 \geq \ldots \geq \lambda_\ell \geq \mu_\ell \geq \lambda_{\ell+1} \geq 0$, and $\psi_{\lambda/\mu} = 0$ otherwise.

- The above recursive relations allows to introduce the corresponding recursive operators.
- $\mathcal{Q}^{\mathfrak{gl}_{n+1}}_{\mathfrak{gl}_n}(\lambda_{n+1})$:

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\lambda_{\ell+1})\cdot f(\underline{x}_{\ell+1})$$

$$= \int_T d^\times\underline{x}_\ell \, Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1};\, \underline{x}_\ell |\, \lambda_{\ell+1})\, \Delta(\underline{x}_\ell)\, f(\underline{x}_\ell^{-1})\,,$$

- Remark.

$$\mathcal{Q}^{\mathfrak{gl}_{n+1}}_{\mathfrak{gl}_n}(\lambda_{n+1}) P^{\mathfrak{gl}_\ell} \sim \mathcal{Q}^{\mathfrak{gl}_\ell} \cdot \check{\mathcal{Q}}^{\mathfrak{gl}_\ell} \cdot P^{\mathfrak{gl}_\ell}$$

$$\prod_{i=1}^{\ell+1}\prod_{j=1}^{\ell} \Gamma_{q,t}(x_{\ell+1,\,i},\, x_{\ell,\,j}) = \prod_{i,j=1}^{\ell} \Gamma_{q,t}(x_{\ell+1,\,i},\, x_{\ell,\,j})\, L^\vee_{x_{\ell+1,\ell+1}}(\underline{x}_\ell)$$

- $\overset{\vee}{\mathcal{Q}}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(x_{\ell+1})$:

$$\overset{\vee}{\mathcal{Q}}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(x_{\ell+1})\cdot f(\underline{\lambda}_{\ell+1}) \;=\; \sum_{\underline{\lambda}_\ell}\overset{\vee}{Q}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{\lambda}_{\ell+1},\,\underline{\lambda}_\ell|\,x_{\ell+1})\,f(\underline{\lambda}_\ell)\,,$$

$$\overset{\vee}{Q}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{\lambda}_{\ell+1},\,\underline{\lambda}_\ell|\,x_{\ell+1}) \;=\; x_{\ell+1}^{|\underline{\lambda}_{\ell+1}|-|\underline{\lambda}_\ell|}\psi_{\underline{\lambda}_{\ell+1}/\underline{\lambda}_\ell}\,,$$

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$$\psi_{\lambda/\mu}=\prod_{1\leq i\leq j\leq\ell}\frac{\Gamma_{q,\,tq^{-1}}\big(t^{j-i}q^{\mu_i-\mu_j+1}\big)}{\Gamma_{q,\,tq^{-1}}\big(t^{j-i}q^{\lambda_i-\mu_j+1}\big)}\frac{\Gamma_{q,\,tq^{-1}}\big(t^{j-i}q^{\lambda_i-\lambda_{j+1}+1}\big)}{\Gamma_{q,\,tq^{-1}}\big(t^{j-i}q^{\mu_i-\lambda_{j+1}+1}\big)}$$

when $\lambda_1\geq\mu_1\geq\ldots\geq\lambda_\ell\geq\mu_\ell\geq\lambda_{\ell+1}\geq 0$, and $\psi_{\lambda/\mu}=0$ otherwise.

- The existence of the two dual recursive representations provide a family of 2^{ℓ} integral representations for the Macdonald polynomials.
- Let $n = 1, \dots, \ell$. Define

$$R^{I}_{n+1,\,n} \,:=\, \mathcal{Q}^{\mathfrak{gl}_{n+1}}_{\mathfrak{gl}_n}(\lambda_{n+1})\,,\, R^{II}_{n+1,\,n} \,:=\, \check{\mathcal{Q}}^{\mathfrak{gl}_{n+1}}_{\mathfrak{gl}_n}(x_{n+1})\,,\,,$$

then for every array $\epsilon = (\epsilon_1, \dots, \epsilon_\ell)$ of $\epsilon_n \in \{I,\, II\}$, $n = 1, \dots, \ell$ the following holds:

$$P^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}_{\ell+1}}(\underline{x}_{\ell+1}) \,=\, \Big\{\, R^{\epsilon_\ell}_{\ell+1,\,\ell} \circ \ldots \circ R^{\epsilon_1}_{2,\,1} \circ R_{1,\,0} \Big\} \cdot 1\,,$$

where $R_{1,\,0}$ is the $\mathfrak{gl}_1$-Macdonald polynomial $P^{\mathfrak{gl}_1}$.

The q-deformed Toda chain on the lattice ($t \longrightarrow \infty$ limit). [GLO]: math.RT/0803.0145; math.RT/0803.0970; math.RT/0805.3754

- Introduced by Ruijsenaars (1990)
- Let $\underline{n}_{\ell+1} = (n_{\ell+1,1}, \ldots, n_{\ell+1,\ell+1})$, $n_{ij} \in \mathbb{Z}$. Assume $q \in \mathbb{C}^*$, $|q| < 1$. Denote

$$T_i f(..., n_{\ell+1,i}, ...) = f(..., n_{\ell+1,i} + 1, ...)$$

The q Toda chain is defined by Hamiltonian

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{n}_{\ell+1}) = \sum_{i=1}^{\ell} (1 - q^{n_{\ell+1,i} - n_{\ell+1,i+1} + 1}) T_i + T_{\ell+1}$$

- There exists commuting set of Hamiltonians $\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}$

Class one q-deformed Whittaker function

- The class one q-deformed Whittaker function is common eigenfunction of commuting set of Hamiltonians:

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{n}_{\ell+1})\Psi_{\underline{z}}^{\mathfrak{gl}_{\ell+1}}(\underline{n}_{\ell+1}) = \Big(\sum_{I_r}\prod_{i\in I_r} z_i\Big)\ \Psi_{\underline{z}}^{\mathfrak{gl}_{\ell+1}}(\underline{n}_{\ell+1})$$

- Weyl invariant over z,
- $\Psi_{\underline{z}}^{\mathfrak{gl}_{\ell+1}}(\underline{n}_{\ell+1}) = 0$ outside dominant domain $n_{\ell+1,1} \geq ... \geq n_{\ell+1,\ell+1}$

- Let us define normalized symmetric polynomials

$$P_\lambda^{qW}(x) = \Delta_q^{-1}(\lambda) P_\lambda(x; q, t=0),$$

$$\Delta_q(\lambda) = \prod_{i=1}^{\ell} (\lambda_i - \lambda_{i+1})_q!,$$

where $(n)_q! = \prod_{i=1}^{n}(1 - q^{n+1-i})$.
- In the following we will call $P_\lambda^{qW}(x)$ the q-Whittaker polynomials. These polynomials were introduced in [Gerasimov, Lebedev and Oblezin] as class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions (see also [Cherednik] for generalization to arbitrary root systems)

Theorem

Let $\lambda_1 \geq ... \geq \lambda_{\ell+1}$. The class one q-Whittaker function coincides with the normalized q-Whittaker polynomial

$$\Psi_x^{\mathfrak{gl}_{\ell+1}}(\lambda) = P_\lambda^{qW}(x)$$

"Pieri type" Q-operator

- Definition

  The Q- operator acting in $\Lambda_q^{(\ell+1)}$ is a family of integral operators

$$\mathbf{Q}_z \cdot f(\lambda) \,=\, \sum_{\mu\in\mathbb{Z}^{\ell+1}} \Delta_q(\mu)\,\mathbf{Q}_{\ell+1,\,\ell+1}(\mu;\,\lambda|\,z)\,f(\mu)\,,$$

  with the kernel

$$\mathbf{Q}_{\ell+1,\,\ell+1}(\mu,\,\lambda|\,z) \,=\, z^{|\mu-\lambda|}\,\varphi^q_{\mu/\lambda} \,=\, z^{|\mu-\lambda|}\,\varphi_{\mu/\lambda}|_{t=0}\Delta_q(\lambda)^{-1}$$

$$= \, z^{|\mu-\lambda|}\,\frac{\Theta(\mu_1-\lambda_1)}{(\mu_1-\lambda_1)_q!}\prod_{i=1}^{\ell}\frac{\Theta(\lambda_i-\mu_{i+1})}{(\lambda_i-\mu_{i+1})_q!}\frac{\Theta(\mu_{i+1}-\lambda_{i+1})}{(\mu_{i+1}-\lambda_{i+1})_q!}\,.$$

Theorem

The action of the Baxter operator $\mathbf{Q}_z$ *on q-Whittaker polynomials is given by*

$$\mathbf{Q}_z \cdot P_\lambda^{qW}(x) \,=\, \mathbf{L}_z(x)\, P_\lambda^{qW}(x)\,,$$

where

$$\mathbf{L}_z(x) \,=\, \prod_{i=1}^{\ell+1} \Gamma_q(zx_i)\,.$$

"Pieri type" formula of q-Whittaker function

- Denote by $\mathcal{P}^{(\ell+1)} \subset \mathbb{Z}^{\ell(\ell+1)/2}$ a subset of integers $n_{k,i}$, $k = 1, \ldots, \ell+1$, $i = 1, \ldots, k$ satisfying the Gelfand-Zetlin conditions $n_{k+1,i} \geq n_{k,i} \geq n_{k+1,i+1}$ for $k = 1, \ldots, \ell$.

Theorem

Class one q-Whittaker function is given in the dominant domain $n_{\ell+1,1} \geq \ldots \geq n_{\ell+1,\ell+1}$ *by*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{n}_{\ell+1}) = \sum_{n_{k,i}\in\mathcal{P}^{(\ell+1)}} \prod_{k=1}^{\ell+1} z_k^{\sum_i n_{k,i}-\sum_i n_{k-1,i}}$$

$$\times \frac{\prod\limits_{k=2}^{\ell}\prod\limits_{i=1}^{k-1}(n_{k,i}-n_{k,i+1})_q!}{\prod\limits_{k=1}^{\ell}\prod\limits_{i=1}^{k}(n_{k+1,i}-n_{k,i})_q!\,(n_{k,i}-n_{k+1,i+1})_q!},$$

and zero otherwise.

$$(n)_q! = (1-q)...(1-q^n)$$

Limit $q \longrightarrow 1$ [GLO]: arXiv:1101.4567

- Theorem

  Let us use the following parametrization

$$q = e^{-\epsilon},\ n_{\ell+1,k} = (\ell + 2 - 2k)m(\epsilon) + \epsilon^{-1}x_{\ell+1,k}\,,\ z_k = e^{\imath\epsilon\lambda_k}\,,$$

where $k = 1, \dots, \ell+1$, $m(\epsilon) = -[\epsilon^{-1}\ln\epsilon]$. *Then*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}) = \lim_{\epsilon\to+0}\left(\epsilon^{\frac{\ell(\ell+1)}{2}}\, e^{\frac{\ell(\ell+3)}{2}A(\epsilon)}\ \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{n}_{\ell+1})\right),$$

where $A(\epsilon) = -\frac{\pi^2}{6}\frac{1}{\epsilon} - \frac{1}{2}\ln\frac{\epsilon}{2\pi}$

- Where

$$\Psi_{\underline{\lambda}}(\underline{x}|\hbar) = \int\limits_{\mathcal{C}} \exp\left(\frac{\imath}{\hbar}\sum_{k=1}^{\ell+1}\lambda_k\left(\sum_{i=1}^{k}T_{k,i} - \sum_{i=1}^{k-1}T_{k-1,i}\right)\right)\times$$

$$\exp\{\sum_{k=1}^{\ell} -\frac{1}{\hbar}\left(\sum_{i=1}^{k} e^{T_{ki}-T_{k+1,i}} + \sum_{i=1}^{k} e^{T_{k+1,i+1}-T_{k,i}}\right)\}\prod_{i,k} dT_{k,i}\,,$$

where we set $x_i = T_{\ell+1,i}, \quad i = 1,\ldots,\ell+1$ and domain of integration $\mathcal{C}$ is slightly deformed subspace $\mathbb{R}^{\ell(\ell+1)/2} \subset \mathbb{C}^{\ell(\ell+1)/2}$ making the integral convergent.
- Givental (1996)–"Pieri type" representation.
- *Comment:* Vinogradov and Tahtadzhyan (1982) $GL(3)$ case and Stade (1990) $GL(2n+1)$. Gerasimov, Lebedev and Oblezin (2007) found connection of Vinogradov and Tahtadzhyan representation with the Givental one .

Baxter operator in Toda chain theory [Gaudin and Pasquier]

- Pieri type $\mathbf{Q}_z$ operator when $q \longrightarrow 1$ is integral operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)$ with the kernel [Gaudin, Pasque]:

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\, \underline{y}|\,\lambda) =$$

$$\exp\left\{ \imath\lambda \sum_{i=1}^{\ell+1}(x_i - y_i) - \sum_{k=1}^{\ell}\left(e^{x_i - y_i} + e^{y_i - x_{i+1}}\right) - e^{x_{\ell+1} - y_{\ell+1}}\right\}$$

- Recursion operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\lambda)$ is integral operator with kernel:

$$Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}, \underline{y}|\lambda) =$$

$$\exp\left\{\imath\lambda\Big(\sum_{i=1}^{\ell+1} x_i - \sum_{i=1}^{\ell} y_i\Big) - \sum_{i=1}^{\ell}\left(e^{x_i - y_i} + e^{y_i - x_{i+1}}\right)\right\}$$

Example

- For $\ell = 1$ we have

$$\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(n_{2,1},n_{2,2}) = \sum_{n_{2,2}\leq n_{1,1}\leq n_{2,1}} \frac{z_1^{n_{1,1}} z_2^{n_{2,1}+n_{2,2}-n_{1,1}}}{(n_{1,1}-n_{2,2})_q!(n_{2,1}-n_{1,1})_q!},$$

$$\Psi_{z_1,z_2}(n_{2,1},n_{2,2}) = 0, \qquad n_{2,2} > n_{2,1}\,,$$

where $(n)!_=(1-q)...(1-q^n)$.

Example

- Using the parametrization

$$q=e^{-\epsilon},\, n_{21}=m(\epsilon)+x_{21}\epsilon^{-1},\, n_{22}=-m(\epsilon)+x_{22}\epsilon^{-1},$$

with $m(\epsilon)=-[\epsilon^{-1}\ln\,\epsilon]$, $z_i=e^{\imath\epsilon\lambda_i}$ we obtain

$$\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(n_{21},n_{22})=\sum_{x_{22}-\epsilon m(\epsilon)\leq x_{11}\leq x_{2,1}+\epsilon\, m(\epsilon)}$$

$$\frac{e^{\imath\lambda_1 x_{11}+\imath\lambda_2(x_{21}+x_{22}-x_{11})}}{((x_{11}-x_{22})/\epsilon+m(\epsilon))_q!\;((x_{21}-x_{11})/\epsilon+m(\epsilon))_q!},$$

where we use the notations $n_{11}=x_{11}/\epsilon$.

Example

- Taking into account

$$\frac{1}{(y/\epsilon+m(\epsilon))_q!}=e^{+\frac{\pi^2}{6}\frac{1}{\epsilon}+\frac{1}{2}\ln\frac{\epsilon}{2\pi}-e^{-y}+O(\epsilon)},$$

we obtain

$$\psi^{\mathfrak{gl}_2}_{\lambda_1,\lambda_2}(x_1,x_2)\;=\lim_{\epsilon\to+0}\;\epsilon\,e^{-\frac{\pi^2}{3}\frac{1}{\epsilon}-\ln\frac{\epsilon}{2\pi}}\;\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(n_{21},n_{22})$$

$$=\int_{\mathbb{R}}dx_{11}\,e^{\imath\lambda_1x_{11}}\,e^{\imath\lambda_2(x_{21}+x_{22}-x_{11})}\,e^{-e^{x_{11}-x_{21}}-e^{x_{22}-x_{11}}}.$$

Main Lemma

- Lemma

Let us introduce the following functions

$$f_\alpha(y,\epsilon) = (y/\epsilon + \alpha m(\epsilon))_q!, \qquad \alpha = 1,2,$$

where $m(\epsilon) = -[\epsilon^{-1}\ln\epsilon]$, $q = e^{-\epsilon}$. *Then for* $\epsilon \to +0$ *the following expansions hold:*

$$f_1(y,\epsilon) = e^{A(\epsilon)+e^{-y}+O(\epsilon)}\,;$$

$$f_2(y,\epsilon) = e^{A(\epsilon)+O(\epsilon^{\alpha-1})},$$

where $A(\epsilon) = -\frac{\pi^2}{6}\frac{1}{\epsilon} - \frac{1}{2}\ln\frac{\epsilon}{2\pi}$.

"Cauchy-Littlewood type" representation [Kharchev, Lebedev]

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}) =$$

$$\int\limits_{\mathcal{S}} \prod_{n=1}^{\ell} \frac{\prod\limits_{k=1}^{n}\prod\limits_{m=1}^{n+1}\Gamma(\imath\gamma_{nk}-\imath\gamma_{n+1,m})}{(2\pi)^n n! \prod\limits_{s\neq p}\Gamma(\imath\gamma_{ns}-\imath\gamma_{np})} e^{-\imath\sum\limits_{n=1}^{\ell+1}\sum\limits_{j=1}^{n}(\gamma_{nj}-\gamma_{n-1,j})x_n} \prod_{\substack{n=1\\ j\leq n}}^{\ell} d\gamma_{nj}\,,$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{\ell+1}) := (\gamma_{\ell+1,1}, \ldots, \gamma_{\ell+1,\ell+1})$

Whittaker function

- $\mathcal{V}_{\underline{\lambda}} = \mathrm{Ind}_B^G \chi_{\underline{\lambda}}$ is a principal series representation of G induced from the generic character $\chi_{\underline{\lambda}}$ of B trivial on $N \subset B$ with $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{\ell+1})$. It is realized in the space of functions $f \in C^\infty(G)$ satisfying equation

$$f(bg) = \chi_{\underline{\lambda}}(b) f(g),$$

where $b \in B$. The action of G is given by the right action $\pi_{\underline{\lambda}}(g) f(x) = f(xg)$

Whittaker function

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g) := e^{-\rho(g)}\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g) = e^{-\rho(g)} < \phi_K, \pi_{\underline{\lambda}}(g)\, \psi_R >,$$

where $\rho(g)$ is given by $\rho(kan) =< \rho, \log a >$, ϕ_K is a spherical vector in $\mathcal{V}_{\underline{\lambda}}$

$$\phi_K(bgk) = \chi_{\underline{\lambda}}(b)\phi_K(g), \qquad k \in K,\ b \in B_+,$$

ψ_R- is eigenvector of simple roots generators of $\mathfrak{g}$

Whittaker function solves Toda chain

- Due to functional equation $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g)$ descends to a function on the space A of the diagonal matrices $a = \mathrm{diag}(e^{\tilde{x}_1}, \ldots, e^{\tilde{x}_{\ell+1}})$ entering the Iwasawa decomposition $KAN_- \longrightarrow GL(\ell+1, \mathbb{R})$:
- **Fact** (Kostant, Semenov-Tian-Shansky) The Whittaker function

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}) = e^{-<\rho,x>} < \phi_K \, , \pi_{\underline{\lambda}}(e^{\sum_{i=1}^{\ell+1} x_i E_{i,i}}) \, \psi_R >$$

is the Toda eigenfunction.

Spherical Hecke algebra

- The spherical Hecke algebra is defined as an algebra of K-biinvariant functions on $G(\mathbb{R})$ (K - maximal compact subgroup of $G(\mathbb{R})$), $\phi(g) = \phi(k_1 g k_2)$, $k_1, k_2 \in K$ acting by a convolution

$$\phi * f(g) = \int_G \phi(g\tilde{g}^{-1})\, f(\tilde{g}) d\tilde{g}.$$

- To ensure the convergence of the integrals one usually imposes the condition of **compact support** on K-biinvariant functions. We will consider slightly more general class of **exponentially decaying** functions

L_∞-factor as Hecke eigenvalue

Let $\phi_{\mathcal{Q}_0(\lambda)}(g)$ be a K-biinvariant function on $G = GL(\ell+1,\mathbb{R})$ given by

$$\phi_{\mathcal{Q}_0(\lambda)}(g) = 2^{\ell+1}|\det g|^{\imath\lambda+\frac{\ell}{2}} e^{-\pi \mathrm{Tr} g^t g}.$$

Theorem

(See [GLO 2007])

$$\left(\phi_{\mathcal{Q}_0(\lambda)} * \Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}\right)(g) = L_\infty(\lambda)\ \Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}(g),$$

where $L_\infty(\lambda)$ is the local Archimedean L-factor

$$L_\infty(\lambda) = \prod_{j=1}^{\ell+1} \pi^{-\frac{\imath\lambda-\imath\gamma_j}{2}} \Gamma\Big(\frac{\imath\lambda-\imath\gamma_j}{2}\Big).$$

Theorem

(See [GLO 2007]) Let $a = diag(e^{x_1}, \dots, e^{x_{\ell+1}}) \in A$. *Then*

$$\phi_{\mathcal{Q}_0(\lambda)} * \Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(a) = \int\limits_{\mathbb{R}^{\ell+1}} \mathcal{Q}_0^{\mathfrak{gl}_{\ell+1}}(\underline{x}, \underline{y}|\lambda) \Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(e^{\underline{y}}) d\underline{y},$$

where

$$\mathcal{Q}_0^{\mathfrak{gl}_{\ell+1}}(\underline{x}, \underline{y}|\lambda) = 2^{\ell+1} \exp\Big\{ \sum_{j=1}^{\ell+1} (\imath\lambda + \rho_j)(x_j - y_j)$$

$$-\pi \sum_{k=1}^{\ell} \Big(e^{2(x_k - y_k)} + e^{2(y_k - x_{k+1})} \Big) - \pi e^{2(x_{\ell+1} - y_{\ell+1})} \Big\},$$

Up to conjugation by $e^{-\rho(x)}$ the $\mathcal{Q}_0^{\mathfrak{gl}_{\ell+1}}$ is the kernel of Baxter operator

Universal Baxter operator for $Sp_{2\ell}$ and $SO_{2\ell}$ and work of Piatetsky-Shapiro and Rallis ([GL 2011]

- **Embedding $Sp_{2\ell} \to GL_{2\ell}$.** Define involution:

$$g \longmapsto g^* := S \cdot J \cdot (g^{-1})^t \cdot J^{-1} \cdot S^{-1},$$

  $a \to a^t$ is standard matrix transposition,

$$S = \mathrm{diag}(1, -1, \dots, -1, 1),$$

  $J = \|J_{i,j}\| = \|\delta_{i+j,2\ell+1}\|$.

- The symplectic group $G = Sp_{2\ell}$ is defined as the subgroup of $GL_{2\ell}$:

$$Sp_{2\ell} = \{g \in GL_{2\ell} : g^* = g\}.$$

- **Embedding $SO_{2\ell} \to GL_{2\ell}$.** Define involution:

$$g \longmapsto g^* := S \cdot J \cdot (g^{-1})^t \cdot J^{-1} \cdot S^{-1},$$

$$S = \mathrm{diag}(1, -1, \ldots, (-1)^{\ell-1}, (-1)^{\ell-1}, (-1)^{\ell}, \cdots, 1),$$

$J = \|J_{i,j}\| = \|\delta_{i+j,2\ell+1}\|$.

- The orthogonal group $G = SO_{2\ell}$ is defined as the subgroup of $GL_{2\ell}$:

$$SO_{2\ell} = \{g \in GL_{2\ell} : g^* = g\}.$$

- The maximal compact subgroup of $G = SO_{2\ell}, Sp_{2\ell}$ embedded this way is given by an intersection of G with the maximal compact subgroup of $GL_{2\ell}(\mathbb{R})$.

- We would like to construct elements of Hecke algebra $\mathcal{H}(G,K)$, where G are maximal split forms of $Sp_{2\ell}$, $SO_{2\ell}$ and $K \subset G$ is a maximal compact subgroup, such that their actions on spherical vectors in spherical principle series representations are given by multiplications on the corresponding local Archimedean L-factors associated with the standard representations of the dual Lie groups. Recall that $SO_{2\ell}$ is self-dual and $Sp_{2\ell}$ is dual to $SO_{2\ell+1}$.
- The corresponding local L-factors are given by

$$L^{SO_{2\ell}}(s,\mu_1,r) = \prod_{i=1}^{\ell} \Gamma_{\mathbb{R}}(s,\lambda_i)\,\Gamma_{\mathbb{R}}(s,-\lambda_i),\, G = SO_{2\ell},$$

$$L^{SO_{2\ell+1}}(s,\mu_2,r) =$$

$$\Gamma_{\mathbb{R}}(s,0)\,\prod_{i=1}^{\ell} \Gamma_{\mathbb{R}}(s,\lambda_i)\,\Gamma_{\mathbb{R}}(s,-\lambda_i),\, G = Sp_{2\ell},\, G^{\vee} = SO_{2\ell+1}$$

Definition

Let $G = SO_{2\ell}$ or $Sp_{2\ell}$. Define the universal Baxter operator $\mathcal{Q}_G(g,s)$ to be a one-parameter family of functions on G given by

$$\mathcal{Q}_G(g,s) = d_G(s)\ \frac{R_G(g,s)}{R_G(0,s)},\ \ g \in G$$

where

$$R_G(g,s) = \int_{GL_{2\ell}(\mathbb{R})} e^{-\pi \mathrm{Tr} Z^t (g^t g + 1) Z}\ |\det Z|^{\imath s}\, dZ,$$

$$d_{SO_{2\ell}}(s) = \prod_{j \equiv 0 (\mathrm{mod} 2), j=0}^{2\ell-2} \Gamma_{\mathbb{R}}(2\imath s - j),$$

$$d_{Sp_{2\ell}}(s) = \prod_{j \equiv 0 (\mathrm{mod} 2), j=2}^{2\ell} \Gamma_{\mathbb{R}}(2\imath s - j).$$

Theorem

The $\mathcal{Q}_G(g,s)$ is a K-biinvariant function on G and

$$\mathcal{Q}_G(s) * \phi_K(g) = \int_G dg_1\, \mathcal{Q}_G(g_1,s)\, \phi_K(gg_1^{-1}) =$$

$$L^{G^\vee}(s,\underline{\mu})\phi_K(g),$$

where ϕ_K is spherical in a principle series representation $\mathcal{V}_{\underline{\mu}} = \mathrm{Ind}_B^G\, \chi_\mu$. And $G^\vee = SO_{2\ell}$ or $SO_{2\ell+1}$ for $G = SO_{2\ell}$ or $Sp_{2\ell}$.

Follows from results of [Piatetsky-Shapiro and Rallis]

Conclusion

- The Baxter operator formalism for the Macdonald polynomials is developed.
- Limits to q- deformed and classical class one Whittaker functions are considered.
- The Baxter operator for open Toda chain is described as restriction to the Cartan torus of the universal Baxter operator considered as an element in the spherical Hecke algebra.
- The universal Baxter operators for classical groups are constructed and connection with the work of Piatetski-Shapiro and Rallis on integral representation of local Archimedean L-factors is established .