Recursion operators in a generalized setting

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Integrability and quantization
Conférence en l'honneur de Michael Semenov-Tian-Shansky
Université de Bourgogne, Dijon
1^{er}-3 Juillet 2013

Some 25 years ago

In 1986, Leningrad came to Paris: Semenov-Tian-Shansky gave a series of lectures at the École Normale Supérieure.
They were eventually published as
"Classical r-matrices, Lax equations, Poisson Lie groups and dressing transformations", Lecture Notes in Phys., 280.

In 1991, Semenov-Tian-Shansky gave lectures at the CIMPA school In memory of Jean-Louis Verdier
in Sophia-Antipolis, "Lectures on R-matrices, Poisson-Lie groups and integrable systems", *Lectures on Integrable Systems*, World Scientific, 1994.

In1996, he gave lectures at the CIMPA school in Pondicherry (India), "Quantum and classical integrable systems", *Integrability of Nonlinear Systems*, Lecture Notes in Phys., 495, 1997, re-edited in 2004 as Lecture Notes in Phys., 638.

Modified Yang–Baxter equation

In 1983, in "What is a classical r-matrix?", STS introduced the MYBE.

Let $R : \mathfrak{g} \to \mathfrak{g}$. Set

$$[u, v]^R = [Ru, v] + [u, Rv].$$

Set

$$B^R(u, v) = [Ru, Rv] - R[u, v]^R.$$

Then $[u, v]^R$ satisfies the Jacobi identity iff B^R satisfies

$$[u, B^R(v, w)] + [v, B^R(w, u)] + [w, B^R(u, v)] = 0.$$

A sufficient condition is that $B^R(u, v)$ be proportioanl to $[u, v]$,

$$\boxed{[Ru, Rv] - R([Ru, v] + [u, Rv]) + \lambda^2[u, v] = 0,}$$

MYBE, with coefficient λ^2.

Recursion operators

Meanwhile the theory of bi-Hamiltonian systems was developped (Magri, 1978, Gelfand and Dorfman, 1979–1982).

Recursion operators were introduced ;

Gelfand and Dorfman, 1979, Theorem 4.2: regular operators

Fokas-Fuchssteiner 1980, 1981: hereditary symmetries

Magri-Morosi 1984: Nijenhuis tensors

Deformations of Lie brackets

Let $N : \mathfrak{g} \to \mathfrak{g}$. Set

$$[u, v]_N = [Nu, v] + [u, Nv] - N[u, v].$$

Set

$$T_N(u, v) = [Nu, Nv] - N[u, v]_N$$

Then $[u, v]_N$ satisfies the Jacobi identity iff T_N satisfies

$$[u, T_N(v, w)] + [v, T_N(w, u)] + [w, T_N(u, v)] = 0.$$

A sufficient condition is that the torsion T_N of N vanish,

$$\boxed{[Nu, Nv] - N([Nu, v] + [u, Nv]) + N^2[u, v] = 0.}$$

The similarity with MYBE is striking, the two equations coincide if N^2 is proportional to the identity.

Outline

We shall consider Nijenhuis operators, which are the recursion operators of integrable Hamiltonian systems. and define Dirac pairs in terms of Nijenhuis relations in generalized geometry.

- Generalized tangent bundles, Dirac structures.
- Relations in sets and in vector bundles.
- Torsion of a relation, Nijenhuis relations.

The aim is to prove that the notion of Dirac pairs unifies and generalizes

Hamiltonian pairs (bi-Hamiltonian structures),
PΩ-structures,
a restricted class of ΩN-structures.

- Examples

Generalizing the notion of Nijenhuis operator

There are many ways to generalize the notion of Nijenhuis operator. I shall present only one such generalization : Nijenhuis relations, that have to be considered when dealing with Dirac pairs.

Dirac pairs were defined by Irene Ya. Dorfman, in the context of complexes over Lie algebras, following her work with Gelfand [1979][1980].

• Irene Ya. Dorfman, "Dirac structures of integrable evolution equations", *Phys. Lett. A*, **125** (1987).

• —, *Dirac Structures and Integrability of Nonlinear Evolution Equations*, 1993.

Some references

• T. Courant, "Dirac manifolds", *Trans. Amer. Math. Soc.* **319** (1990).

• yks and V. Rubtsov, "Compatible structures on Lie algebroids and Monge-Ampère operators", *Acta. Appl. Math.* **109** (2010).

• yks, "Dirac pairs", *J. Geom. Mech.*, **4** (2012).

• yks, "Nijenhuis structures on Courant algebroids", *Bull. Braz. Math. Soc.*, **42** (2011).

Relations

When U, V and W are sets, the *composition*, $\mathbf{R}' * \mathbf{R}$, of relations $\mathbf{R} \subset U \times V$ and $\mathbf{R}' \subset V \times W$ is

$$\mathbf{R}' * \mathbf{R} = \{(u, w) \in U \times W \,|\, \exists v \in V, (u, v) \in \mathbf{R} \,\mathrm{and}\, (v, w) \in \mathbf{R}'\}.$$

The *transpose* of a relation $\mathbf{R} \subset U \times V$ is the relation

$$\overline{\mathbf{R}} = \{(v, u) \in V \times U \,|\, (u, v) \in \mathbf{R}\}.$$

If $\phi : U \to V$ and $\phi' : V \to W$ are maps, and if $\mathbf{R} = \mathrm{graph}\,\phi$ and $\mathbf{R}' = \mathrm{graph}\,\phi'$, then

$$\mathbf{R}' * \mathbf{R} = \mathrm{graph}(\phi' \circ \phi).$$

If $\phi : U \to V$ is invertible,

$$\overline{\mathrm{graph}\,\phi} = \mathrm{graph}(\phi^{-1}).$$

Relations in vector spaces and vector bundles

Let U and V be vector spaces. The *dual* of a relation $\mathbf{R} \subset U \times V$ is the relation $\mathbf{R}^* \subset V^* \times U^*$ defined by

$$\mathbf{R}^* = \{(\beta, \alpha) \in V^* \times U^* \mid \langle \alpha, u \rangle = \langle \beta, v \rangle, \forall (u, v) \in \mathbf{R}\}.$$

If $\mathbf{R} = \mathrm{graph}\, \phi$, where ϕ is a linear map from U to V, then $\mathbf{R}^*$ is the graph of the dual map, ϕ^*.

Convention When U and V are vector bundles over a manifold M, and $\mathbf{R} \subset U \times V$ is a relation, we denote by the same letter the relation on sections induced by $\mathbf{R}$.

Generalized tangent bundles

The generalized tangent bundle of a smooth manifold, M, is

$$\mathbf{T}M = TM \oplus T^*M$$

equipped with

- the canonical fibrewise non-degenerate, symmetric, bilinear form

$$\boxed{\langle X+\xi, Y+\eta\rangle = \langle X, \eta\rangle + \langle Y, \xi\rangle,}$$

- the Dorfman bracket

$$\boxed{[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - i_Y(d\xi),}$$

X, Y vector fields, sections of TM, ξ, η differential 1-forms, sections of T^*M.

The Dorfman bracket is a derived bracket, $i_{[X,\eta]} = [[i_X, d], e_\eta]$.

For derived brackets, see yks, Ann. Fourier 1996, LMP 2004.

Properties of the Dorfman and Courant brackets

The Dorfman bracket is not skew-symmetric, it is a Loday (Leibniz) bracket, *i.e.*, it satisfies the Jacobi identity in the form

$$\boxed{[u,[v,w]] = [[u,v],w] + [v,[u,w]],}$$

when u and v are sections of $\mathbf{T}M = TM \oplus T^*M$.

The Courant bracket is the skew-symmetrized Dorfman bracket,

$$\boxed{[X+\xi, Y+\eta] == [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \frac{1}{2}\langle X+\xi, Y+\eta\rangle.}$$

The Courant bracket is skew-symmetric but it does not satisfy the Jacobi identity.

The generalized tangent bundle of M, $\mathbf{T}M$, is the double of TM. It is a Courant algebroid.

More generally, the double of a Lie bialgebroid is a Courant algebroid.

Lie algebroids

Definition: A Lie algebroid is a vector bundle $\tau : A \to M$ such that

• ΓA is a Lie algebra over $\mathbb{R}$
skewsymmetry + Jacobi identity)

• there exists a morphism of vector bundles $\rho : A \to TM$, called the anchor, such that the Leibniz identity is satisfied,

$$\boxed{\forall X, Y \in \Gamma A, \forall f \in C^\infty(M), [X, fY] = f[X, Y] + (\rho(X) \cdot f)Y}$$

Proposition The mapping ρ induces a Lie algebra homomorphism $\Gamma A \to \Gamma(TM)$.

Examples

- Lie algebras
- TM
- foliations: integrable subbundles of TM
- cotangent bundle of a Poisson manifold, (M, π)
  there exists a unique Lie algebroid structure $[\ ,\ ]_\pi$ on T^*M such that
  the anchor is the map $\pi^\sharp : T^*M \to TM$ defined by π, and

$$[df, dg]_\pi = d\{f, g\}, \quad \forall f, g \in C^\infty(M).$$

- gauge Lie algebroids (also called Atiyah algebroids): TP/G, where P is a principal bundle with structure group G.
- action Lie algebroids : $\mathfrak{g} \times M$, where $\mathfrak{g}$ is a Lie algebra acting on M.

The Schouten–Nijenhuis bracket

If A is a Lie algebroid there is an odd Poisson (Gerstenhaber) bracket on $\Gamma(\wedge^\bullet A)$ that generalizes the Schouten–Nijenhuis bracket of multivector fields.
It is a bracket denoted by $[\ ,\ ]_{SN}$ or just $[\ ,\ ]$, of degree -1 on the space of sections of $\Gamma(\wedge^\bullet A)$, the unique extension to $\Gamma(\wedge^\bullet A)$ as a (graded) bi-derivation of the Lie bracket of vector fields satisfying $[X, f] = X \cdot f$, for all $X \in \Gamma(A)$, $f \in C^\infty(M)$.

Lie bialgebroids

A Lie bialgebroid is defined by a pair of Lie algebroids in duality with a compatibility condition.
Lie bialgebroids generalize the Lie bialgebras.

Standard example:
(TM, T^*M) where M is a Poisson manifold and T^*M is equipped with the Lie bracket of 1-forms

If (A, A^*) is a Lie bialgebroid, there is a Dorfman bracket on $A \oplus A^*$.

The generalized tangent bundle is the particular case where T^*M has a vanishing Lie bracket.

Dirac structures

A sub-bundle $L \in TM \oplus T^*M$ is a Dirac structure if it is maximally isotropic and closed under the Dorfman bracket.

• If $\pi : T^*M \to TM$, π is a Poisson structure iff its graph is a Dirac structure in $TM \oplus T^*M$

• If $\omega : TM \to T^*M$, ω is a presymplectic structure ($\mathrm{d}\omega = 0$) iff its graph is a Dirac structure in $TM \oplus T^*M$.

Same definition in $A \oplus A^*$.

Torsion of a relation

Let **N** be a relation in $E \times E$, where $(E, [\ ,\ ])$ is a Loday algebra. Consider the real-valued function defined on a subset of $E \times E \times E \times E \times E^* \times E^* \times E^*$ by

$$\mathbf{T}(\mathbf{N})(u_1, v_1, u_2, v_2, \alpha, \alpha', \alpha'') \\ = \langle \alpha, [v_1, v_2] \rangle - \langle \alpha', [v_1, u_2] + [u_1, v_2] \rangle + \langle \alpha'', [u_1, u_2] \rangle,$$

for all $u_1, v_1, u_2, v_2 \in E, \alpha, \alpha', \alpha'' \in E^*$ such that $(u_1, v_1) \in \mathbf{N}, (u_2, v_2) \in \mathbf{N}$, $(\alpha, \alpha') \in \mathbf{N}^*, (\alpha', \alpha'') \in \mathbf{N}^*$.

The function $\mathbf{T}(\mathbf{N})$ is called the torsion of the relation **N**.

Definition

A Nijenhuis relation in $E \times E$ is a subset **N** of $E \times E$ such that its torsion, $\mathbf{T}(\mathbf{N})$, vanishes.

Nijenhuis relations generalize Nijenhuis tensors

Proposition

Let $(E, [\ ,\ ])$ *be a Loday algebra. A linear map,* $N : E \to E$*, is a* *Nijenhuis tensor* *if and only if* $\mathrm{graph}\, N$ *is a* *Nijenhuis relation* *in* $E \times E$*.*

Dirac pairs

Let A be a vector bundle, and let A^* be the dual vector bundle. For relations $L \subset A \times A^*$ and $L' \subset A \times A^*$, we consider the relation in $A \times A$,

$$\boxed{\mathbf{N}_{L,L'} = \overline{L} * L'.}$$

Assume that A is a Lie algebroid, and that $E = A \oplus A^*$ is equipped with the Dorfman bracket.

Definition

Dirac structures L and L' on A are said to be a *Dirac pair* if $\mathbf{N}_{L,L'}$ is a Nijenhuis relation in $A \times A$.

Hamiltonian pairs

Let A be a Lie algebroid.

Lemma.

A bivector π is a Poisson structure on A if and only if, for all $\xi_1, \xi_2 \in \Gamma(A^*)$,

$$\boxed{[\pi\xi_1, \pi\xi_2] = \pi[\xi_1, \xi_2]_\pi,}$$

where $[\ ,\ ]_\pi$ is the bracket of sections of A^* defined by π,

$$[\xi_1, \xi_2]_\pi = L_{\pi\xi_1}\xi_2 - L_{\pi\xi_2}\xi_1 + d(\pi(\xi_1, \xi_2)).$$

Definition

Poisson structures π and π' on A are said to be *compatible* if $\pi + \pi'$ is a Poisson structure. When Poisson structures π and π' are compatible, (π, π') is said to be a *bi-Hamiltonian structure* or a *Hamiltonian pair*.

Fact: Poisson structures π and π' constitute a Hamiltonian pair if and only if $[\pi, \pi'] = 0$, where $[\ ,\ ]$ is the Schouten–Nijenhuis bracket.

The relation defined by a pair of bivectors

For bivectors π and π', set

$$\boxed{\mathbf{N}(\pi, \pi') = \operatorname{graph} \pi * \overline{\operatorname{graph} \pi'}.}$$

Theorem

Let π and π' be bivectors. The torsion of the relation $\mathbf{N}(\pi, \pi')$ satisfies the equation

$$2\mathbf{T}(\mathbf{N}(\pi, \pi'))(\xi_1, \xi_2, \xi, \xi', \xi'')$$

$$\langle \xi, [\pi, \pi](\xi_1, \xi_2) \rangle + \langle \xi'', [\pi', \pi'](\xi_1, \xi_2) \rangle - 2\langle \xi', [\pi, \pi'](\xi_1, \xi_2) \rangle.$$

for all $\xi_1, \xi_2, \xi, \xi', \xi'' \in \Gamma(A^*)$ such that $\pi\xi = \pi'\xi'$ and $\pi\xi' = \pi'\xi''$.

Proof Use $[\pi\xi_1, \pi\xi_2] = \pi[\xi_1, \xi_2]_\pi$ and the skew-symmetry of π and π'. $\square$

Hamiltonian pairs and Poisson pairs

Corollary

If (π, π') is a Hamiltonian pair, then $\mathbf{N}(\pi, \pi')$ is a Nijenhuis relation.

Let us call Poisson bivectors π and π' on A such that $\mathbf{N}(\pi, \pi')$ is a Nijenhuis relation a *Poisson pair*. Then we can state:

Any Hamiltonian pair is a Poisson pair.

In order to state a converse, let us set

$$K = \pi^{-1}(\mathrm{Im}\,\pi') \cap \pi'^{-1}(\mathrm{Im}\,\pi) \subset A^*.$$

Corollary

(i) If (π, π') is a Poisson pair, then $i_\xi[\pi, \pi'] = 0$ for all $\xi \in K$.
(ii) If, in addition, $K = A^*$, then (π, π') is a Hamiltonian pair.

In particular,

Any non-degenerate Poisson pair is a Hamiltonian pair.

(Non-degenerate means that both bivectors are non-degenerate.)

Hierarchies of Poisson structures

The preceding results imply the well known proposition [Fuchssteiner-Fokas, Dorfman, yks-Magri, etc.],

Proposition

(i) Assume that (π, π') *is a Hamiltonian pair, where* π *is non-degenerate. Then* $N = \pi' \pi^{-1}$ *is a Nijenhuis tensor.*
(ii) Assume that π *and* π' *are non-degenerate Poisson structures and that* $N = \pi' \pi^{-1}$ *is a Nijenhuis tensor. Then* (π, π') *is a Hamiltonian pair. More generally, all* $(N^k \pi, N^\ell \pi)$ *(*$k, \ell \in \mathbb{N}$*) are Hamiltonian pairs.*

Poisson pairs and Dirac pairs

If $L = \overline{\mathrm{graph}\,\pi}$ and $L' = \overline{\mathrm{graph}\,\pi'}$, then

$$\mathbf{N}_{L,L'} = \mathrm{graph}\,\pi * \overline{\mathrm{graph}\,\pi'} = \mathbf{N}(\pi, \pi').$$

Theorem

(i) Bivectors π and π' constitute a Poisson pair if and only if their graphs constitute a Dirac pair.
(ii) If (π, π') is a Hamiltonian pair, then $(\overline{\mathrm{graph}\,\pi}, \overline{\mathrm{graph}\,\pi'})$ is a Dirac pair.
(iii) Conversely, if $(\overline{\mathrm{graph}\,\pi}, \overline{\mathrm{graph}\,\pi'})$ is a Dirac pair and if π and π' are non-degenerate bivectors, then (π, π') is a Hamiltonian pair.

Presymplectic pairs

Definition

If ω and ω' are presymplectic structures whose graphs constitute a Dirac pair, (ω, ω') is called a *presymplectic pair*. If, in addition, ω and ω' are non-degenerate, (ω, ω') is called a *symplectic pair*.

For $L = \mathrm{graph}\,\omega$, $L' = \mathrm{graph}\,\omega'$,

$$\mathbf{N}_{L,L'} = \overline{\mathrm{graph}\,\omega} * \mathrm{graph}\,\omega'.$$

Theorem

Symplectic pairs are in one-to-one correspondence with non-degenerate Poisson pairs.

Examples from the theory of Monge-Ampère operators

See Kushner–Lychagin–Rubtsov [2007] and Lychagin–Rubtsov–Chekalov [1993]. See yks–Roubtsov [2010].

Let $M = T^*\mathbb{R}^2$ and let Ω be the canonical symplectic form on M. Here $A = TM$. In canonical coordinates (q^1, q^2, p_1, p_2) on M, $\Omega = \mathrm{d}q^1 \wedge \mathrm{d}p_1 + \mathrm{d}q^2 \wedge \mathrm{d}p_2$.
Examples of presymplectic pairs (Ω, ω) are defined by

$$\omega = \omega_H = \mathrm{d}q^1 \wedge \mathrm{d}p_1 - \mathrm{d}q^2 \wedge \mathrm{d}p_2,$$

$$\omega = \omega_E = \mathrm{d}q^1 \wedge \mathrm{d}p_2 - \mathrm{d}q^2 \wedge \mathrm{d}p_1,$$

$$\omega = \omega_P = \mathrm{d}q^1 \wedge \mathrm{d}p_2.$$

The pair (Ω, ω_E) is a 'conformal symplectic couple' as defined by Geiges (Duke [1996], 4-manifolds), *i.e.*, it is a closed, effective 2-form ($\Omega \wedge \omega = 0$), with Pfaffian equal to 1 ($\omega \wedge \omega = \Omega \wedge \Omega$).

PΩ-structures

Definition

A bivector π and a 2-form ω define a *PΩ-structure* on a Lie algebroid A if π is a Poisson bivector, and both ω and ω_N are closed, where $N = \pi \circ \omega$ and $\omega_N = \omega \circ N$.

Proposition

Let π be a Poisson bivector and let ω be a presymplectic form. Then $(\overline{\mathrm{graph}\,\pi}, \mathrm{graph}\,\omega)$ is a Dirac pair if and only if $\pi \circ \omega$ is a Nijenhuis tensor.

Proof If $L = \overline{\mathrm{graph}\,\pi}$ and $L' = \mathrm{graph}\,\omega$, then

$$\mathbf{N}_{L,L'} = \mathrm{graph}\,(\pi \circ \omega).$$

□

Dirac pairs and PΩ-structures

Theorem

(i) If a Poisson structure π and a presymplectic structure ω constitute a PΩ-structure, their graphs constitute a Dirac pair.
(ii) Conversely, if the graphs of a Poisson structure π and a presymplectic structure ω constitute a Dirac pair, and if π is non-degenerate, then π and ω constitute a PΩ-structure.

ΩN-structures

Let N be a $(1,1)$-tensor and ω a 2-form on A such that $\omega \circ N = N^* \circ \omega$. Then ω_N defined by $\omega_N = \omega \circ N$ is a 2-form.

Definition

A 2-form ω and a $(1,1)$-tensor N define an *ΩN-structure* on a Lie algebroid A if $\omega \circ N = N^* \circ \omega$, N is a Nijenhuis tensor, and both ω and ω_N are closed, where $\omega_N = \omega \circ N$.

Examples

In the notation of the previous example, in coordinates on $T^*\mathbb{R}^2$, (q^1, q^2, p_1, p_2), let $N_H = \Omega^{-1} \circ \omega_H$ and $N_E = \Omega^{-1} \circ \omega_E$, so that

$$N_H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad N_E = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then (Ω, N_H) and (Ω, N_E) are ΩN-structures on $T^*\mathbb{R}^2$, with $N_H^2 = \mathrm{Id}$ and $N_E^2 = -\mathrm{Id}$. Thus N_E is a complex structure, and N_H is a product structure on $T^*(\mathbb{R}^2)$.

Let $N_P = \Omega^{-1} \circ \omega_P$, so that $N_P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then (Ω, N_P) is an ΩN-structure with $N_P^2 = 0$, so that N_P is a tangent structure.

The non-degenerate case

Proposition

Let ω be a non-degenerate 2-form and N a $(1,1)$-tensor such that $\omega_N = \omega \circ N$ is skew-symmetric. Then (ω, N) is an ΩN-structure if and only if $(\mathrm{graph}\,\omega, \mathrm{graph}\,\omega_N)$ is a Dirac pair.

Proof When $L = \mathrm{graph}\,\omega$ and $L' = \mathrm{graph}\,\omega_N$,

$$\mathbf{N}_{LL'} = \{(x, y) \in A \times A \,|\, \omega_N x = \omega y\}.$$

Therefore, when ω is invertible, $\mathbf{N}_{LL'} = \mathrm{graph}\, N$.

Example The pairs $(\mathrm{graph}\,\Omega\ ,\ \mathrm{graph}\,\omega_H)$, $(\mathrm{graph}\,\Omega\ ,\ \mathrm{graph}\,\omega_E)$ and $(\mathrm{graph}\,\Omega, \mathrm{graph}\,\omega_P)$ are the Dirac pairs associated with the ΩN-structures described in the previous example.

Weak ΩN-structures

In the next theorem (yks [2011]), the 2-form ω is not assumed to be non-degenerate. Cf. also Dorfman [1993].
Let ω be a 2-form and N a $(1,1)$-tensor such that $\omega_N = \omega \circ N$ is skew-symmetric.
We shall call (ω, N) a *weak ΩN-structure* if ω and ω_N are closed 2-forms, and the torsion of N takes values in the kernel of ω.

We set $\mathbf{N} = \mathbf{N}_{LL'} = \{(x, y) \in A \times A \,|\, \omega_N x = \omega y\}$ and

$$\mathbf{N}^+ = \{(\omega x, \omega_N x) \in A^* \times A^* \,|\, x \in A\}.$$

The relation $\mathbf{N}^+$ is the restriction of the graph of N^* to the image of ω, and a subset of $\mathbf{N}^*$.

Dirac pairs and ΩN-structures

Theorem

(i) If (ω, N) *is an* *ΩN-structure, and if* $\mathbf{N}^+ = \mathbf{N}^*$, *then* $(\mathrm{graph}\,\omega, \mathrm{graph}\,\omega_N)$ *is a Dirac pair.*
(ii) Conversely, if $(\mathrm{graph}\,\omega, \mathrm{graph}\,\omega_N)$ *is a Dirac pair, then* (ω, N) *is a weak ΩN-structure.*

Proof Evaluate $\mathrm{d}\omega$, $\mathrm{d}\omega_N$ and $\mathrm{d}\omega_{N^2}$ on well chosen triples of vectors [...]. □

More generally, all 2-forms $\omega \circ N^2$, $\omega \circ N^3$, $\ldots$, $\omega \circ N^p, \ldots$ are closed. Whence a hierarchy of Dirac pairs.

This property is the basis of the construction of a sequence of integrals in involution for bi-Hamiltonian systems, and for the extension of this property to systems associated to a Dirac pair.

Conclusion

• Generalized geometry appears more and more frequently in the physics literature : supergravity in terms of "generalized connections": Gabella et al., "Type IIB supergravity and generalized complex geometry" (2010), Daniel Waldram, sigma-models, integrable systems. See in particular Barakat–De Sole–Kac (2009).

Bon anniversaire !

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