Spectral curves in gauge/strings duality. Integrability, singular sectors and regularization.

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joint work with L. Martinez Alonso
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Поздравляю, Миша!

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Cachazo, Dijkgraaf, Intriligator, Vafa

spectral curves

(2001, 2002)

. Class of hyperelliptic curves

$$y^2 = W'(z)^2 + f(z)$$

$W(z)$ and $f(z)$ - polynomials

$$\deg f = \deg W - 2$$

- description of low energy dynamics of $N=1$ $U(n)$ su.sy. gauge theories ...

via duality between su.sy. gauge theories and string theories on deformed 3-dim. Calabi-Yau manifolds defined by

$$W'(z)^2 + f(z) + u^2 + v^2 + w^2 = 0$$

process of Geometric Transition

Other appearance:

1. Subclass of Seiberg-Witten curve

$$y^2 = W'(z)^2 + \Phi(z)$$

$\deg W' = N_c$, $\deg \Phi = N_f$ $\quad N_f < 2N_c$

CDIV curves $\quad N_f = N_c - 1$.

2. Asymptotic eigenvalue distribution for matrix models with

$$Z_n = \int e^{-n \operatorname{tr}(W(M))} dM$$

as $n \to \infty$.

3. Asymptotic distribution of zeros for orthogonal pol. $P_n(z) = z^n + \ldots$ with

$$\int_\Gamma P_n(z)\, z^k e^{-nW(z)} dz = 0, \quad k = 0, 1, \ldots, n-1$$

as $n \to \infty$

4. ...

CDIV curves have

$2q$ $(1 \le q \le N)$ branch points

- odd-order roots $\vec{\beta} = (\beta_1, \dots, \beta_{2q})$ of pol. $y^2(z)$.

q cuts $\gamma_j = [\beta_{2j-1}, \beta_{2j}]$ $j = 1, \dots, q$.

at $f(z) = 0$ cuts shrink into points (double roots). $W'(q_j) =$ [illegible]

Important characteristics:

't Hooft parameters $\vec{S} = (S_1, \dots, S_q)$

defined by (period relations)

$$\oint_{A_j} y(z)\,dz = -4\pi i S_j, \qquad j = 1, \dots, q$$

where A_j - counterclockwise cycles around γ_j

and $y(z) = W'(z) + O\left(\frac{1}{z}\right)$ as $z \to \infty$

Free energy

$$F = \int_\gamma \rho(z) W(z) |dz| - \frac{1}{2} \int_\gamma \int_\gamma \rho(z) \rho(z') \log(z - z')^2 |dz| |dz'|$$

density $\rho(z)$

$$\rho(z)|dz| = \frac{y(z_+)\,dz}{2\pi i} = -\frac{y(z_-)\,dz}{2\pi i}, \quad z \in \gamma$$

Study of moduli space for CDIV curves.

$$W(z) = \frac{z^{N+1}}{N+1} + \sum_{n=1}^{N} g_n z^n,$$

$$f(z) = \sum_{k=1}^{N} t_k z^{N-k}$$

g_k - fixed, $\vec{t} = (t_1, \ldots, t_N)$ - deformation parameters

Study of analytic properties of roots for $y^2(z)$

$\Rightarrow$ analytical properties of density ρ and free energy $\mathcal{F}$

as the functions of S_j and t_k. !

In general

$$y^2(z) = \prod_{e=1}^{p} (z-\alpha_e)^{2p_e} \prod_{i=1}^{2q} (z-\beta_i)^{2q_i+1}$$

$p_e \geq 1$
$q_i \geq 0.$

with $N = \sum_{e=1}^{p} p_e + \sum_{i=1}^{2q} q_i + q$

$p + 2q - N$ independent parameters

Subclass of CDIV curves completely defined by q t'Hooft parameters

$$y^2(z) = \prod_{l=1}^{p} (z-\alpha_l)^2 \prod_{i=1}^{2q} (z-\beta_i)$$

$$p+q = N.$$

Definition.

$\mathcal{M}_q$ - set of all CDIV curves which have $2q$ branch points (q cuts) at most.

$q = 0, 1, \dots, N$ - given.

$$\mathcal{M}_0 \subset \mathcal{M}_1 \cdots \subset \mathcal{M}_N.$$

Definition.

Regular sector $\mathrm{reg}\mathcal{M}_q$ - subset of all curves in $\mathcal{M}_q$ with all roots (α_l, β_i) distinct.

Singular sector

$$\mathrm{sing}\mathcal{M}_q = \mathcal{M}_q / \mathrm{reg}\mathcal{M}_q.$$

Whitham equations

Reduced curve

$$y_{red}^2 = \prod_{i=1}^{2g} (z - \beta_i) \qquad \mathcal{M}_g.$$

One has

$$\left(\frac{W'(z)}{y_{red}(z)} \right)_{\oplus} = \prod_{\ell=1}^{p} (z - \alpha_\ell)$$

$\Rightarrow \quad \alpha_\ell = \alpha_\ell(\beta_1, \dots, \beta_{2g})$

Equations for β_i !

Standard approach.

I. Krichever (1994),

...

L. Chekhov et al (2003) ...

...

Riemann surfaces, Abelian differentials, function $\vartheta(z, \dots)$, ...

Meromorphic differential

$$dS = \frac{1}{2}\left(y(z) + W'(z)\right)dz \qquad \text{on } M_q.$$

where $y(z) = W'(z) - \frac{2S}{z} + O(z^{-2})$, $\quad S = \sum_{i=1}^{q} S_i$

One can show that

$$dS = \sum_{n=1}^{N+1} g_n \, d\Omega_n - \sum_{j=1}^{q} S_j \left(d\Omega_0 + 2\pi i (1-\delta_{jq}) d\varphi_j\right).$$

where

$$d\varphi_j = \frac{P_j(z)}{y_{red}(z)} dz, \quad d\Omega_n = \left(\frac{n}{2} z^{n-1} + \frac{P_n(z)}{y_{red}(z)}\right) dz, \quad d\Omega_0 = \frac{P_0(z)}{y_{red}} dz$$

- Abelian diff. of first, second and third kind.

$$\Rightarrow \frac{\partial\, dS}{\partial S_j} = -d\Omega_0 - 2\pi i (1-\delta_{jq}) d\varphi_j. \Rightarrow$$

$$\frac{\partial y(z)}{\partial S_j} = -2 \frac{P_0(z) - 2\pi i (1-\delta_{jq}) P_j(z)}{y_{red}(z)} \Rightarrow$$

$$\boxed{\frac{\partial \beta_i}{\partial S_j} = 4 \frac{P_0(\beta_i, \vec{\beta}) - 2\pi i (1-\delta_{jq}) P_j(\beta_i, \vec{\beta})}{\prod_{\ell=1}^{p} (\beta_i - \alpha_\ell) \prod_{k \neq i} (\beta_i - \beta_k)}}$$

Prepotential, τ- function.

CDIV curves in terms of deformation parameters.

Curves in $\mathcal{M}_g$ satisfy $2N$ equations:

$$P(\alpha_\ell, t) = P'(\alpha_\ell, t) = 0, \quad \ell = 1, \dots, p$$

$$P(\beta_i, t) = 0, \quad i = 1, \dots, 2g.$$

where

$$P(z, t) = y^2(z, t) = W'(z)^2 + \sum_{k=1}^{N} t_k z^{N-k}.$$

Jacobian of the system

$$\frac{\partial(\quad)}{\partial(\quad)} \dots$$

- nonsingular minors

From implicit function theorem $\to$

Theorem. in reg $\mathcal{M}_g$ each set of g parameters $(t_1, \dots, t_g)$ is an analytic system of coordinates, i.e. all the variables $(\alpha_\ell, \beta_i, t_k)$ are local analytic functions of $(t_1, \dots, t_g)$.

Eliminating (α_0, β_i) from above system, one gets $N-g$ constraints

$$f_k(\bar{t}) = 0, \quad k = 1, \dots, N-g$$

which characterized $\mathcal{M}_g$ in the space $\mathbb{C}^N$ of deformation parameters.

Class $\mathcal{M}_N$ contains all CDIV curves.

In reg $\mathcal{M}_N$

$$\frac{\partial S_j}{\partial t_k} = -\frac{1}{8\pi i} \oint_{A_j} \frac{z^{N-k}}{\sqrt{\prod_{j=1}^{2N}(z-\beta_j)}} dz \qquad j, k = 2, \dots, N$$

$\Rightarrow$ Jacobian of trans: $(t_2, \dots, t_N) \to (S_2, \dots S_N)$ is non singular (it is the matrix of A-periods for Abelian differentials)

$\Rightarrow$

Change of variable $(\bar{t}) \to (\bar{S})$ is an analytic diffeomorphism locally in reg $\mathcal{M}_g$.

Whitham equations for branch points w.r.t. deformation parameters.

$$\beta_i = \beta_i(t_1, \ldots, t_q)$$

From $P(z,t) = y^2(z,t) = W'(z)^2 + \sum_{k=1}^{N} t_k z^{N-k}$

one gets

$$t_{\ell+q} = \sum_{m=1}^{p} V_{\ell m}(\vec{\alpha}) \left[P(\alpha_m, \vec{t}) - \left(W'(\alpha_m)^2 + \sum_{j=1}^{q} t_j \cdot \alpha_m^{N-j} \right) \right]$$

$$\ell = 1, \ldots, p = N - q$$

and $V_{\ell m}(\alpha)$ – inverse of Vandermonde matrix $\left(\alpha_\ell^{p-m}\right)_{\ell,m=1}^{p}$.

$\Rightarrow$

$$P(\beta_i, \vec{t}) - \sum_{\ell,m=1}^{p} \beta_i^{p-\ell} V_{\ell m}(\vec{\alpha}) P(\alpha_m, \vec{t})$$

do not depend on $(t_{q+1}, \ldots, t_N)$.

$$= 0$$

Differentiating above equations w.r.t. $t_1, \dots, t_q$ and taking into account $P(\alpha_l, T) = P'(\alpha_l, T) = 0$, one gets system

$$\frac{\partial \beta_i}{\partial t_j} = \frac{\sum_{l,m} \beta_i^{p-l} \nabla_{lm}(T) \alpha_m^{N-j} - \beta_i^{N-j}}{\prod_l (\alpha_l - \beta_i)^2 \prod_{j \neq i} (\beta_i - \beta_j)}$$

$$i = 1, \dots, 2q$$
$$j = 1, \dots, q.$$

Examples:

$q = N$

$$\frac{\partial \beta_i}{\partial t_j} = - \frac{\beta_i^{N-j}}{\prod_{j \neq i} (\beta_i - \beta_j)}.$$

$q = N-1$

$$\frac{\partial \beta_i}{\partial t_j} = \frac{\alpha^{N-j} - \beta_i^{N-j}}{(\alpha - \beta_i)^2 \prod_{j \neq i} (\beta_i - \beta_j)},$$

$$\alpha = -N g_N - \frac{\beta_1 + \dots + \beta_{2q}}{2}.$$

Cubic model $(N=2)$.

$$W(z) = \frac{z^3}{3} - g z, \qquad g \in \mathbb{C}/\{0\}$$

and

$$y^2(z,t) = (z^2 - g)^2 + t_1 z + t_2, \qquad t_1 = -4S$$

Generic class $\mathcal{M}_2$

$$y^2(z,t) = \prod_{i=1}^{4} (z - \beta_i)$$

In terms of variables $u_1, u_2, \delta_1, \delta_2$ defined as

$$\beta_1 = u_1 + \delta_1, \quad \beta_2 = u_1 - \delta_1, \quad \beta_3 = u_2 + \delta_2, \quad \beta_4 = u_2 - \delta_2.$$

one key equation

$$4u_1^6 - 4g u_1^4 - t_2 u_1^2 - S^2 = 0$$

and

$$u_2 = -u_1, \quad \delta_1^2 = g - u_1^2 + \frac{S}{u_1}, \quad \delta_2^2 = g - u_1^2 - \frac{S}{u_1}$$

that defines $\beta_i(S, t_2)$ $i = 1,2,3,4$.

Regular sector reg$\mathcal{M}_2$

$(s, t_2) \in \mathbb{C}^2$ with

$$27s^4 + (18gt_2 + 16g^3)s^2 - t_2^3 - g^2t_2^2 \neq 0.$$

Sing$\mathcal{M}_2$ $= \mathcal{M}_1$.

for $\mathcal{M}_1$.

$$y^2(z, t) = (z-\alpha)^2(z-\beta_1)(z-\beta_2)$$

In terms of u, δ, $\beta_1 = u + \delta$, $\beta_2 = u - \delta$.
one has

$$u^3 - gu + s = 0$$

and

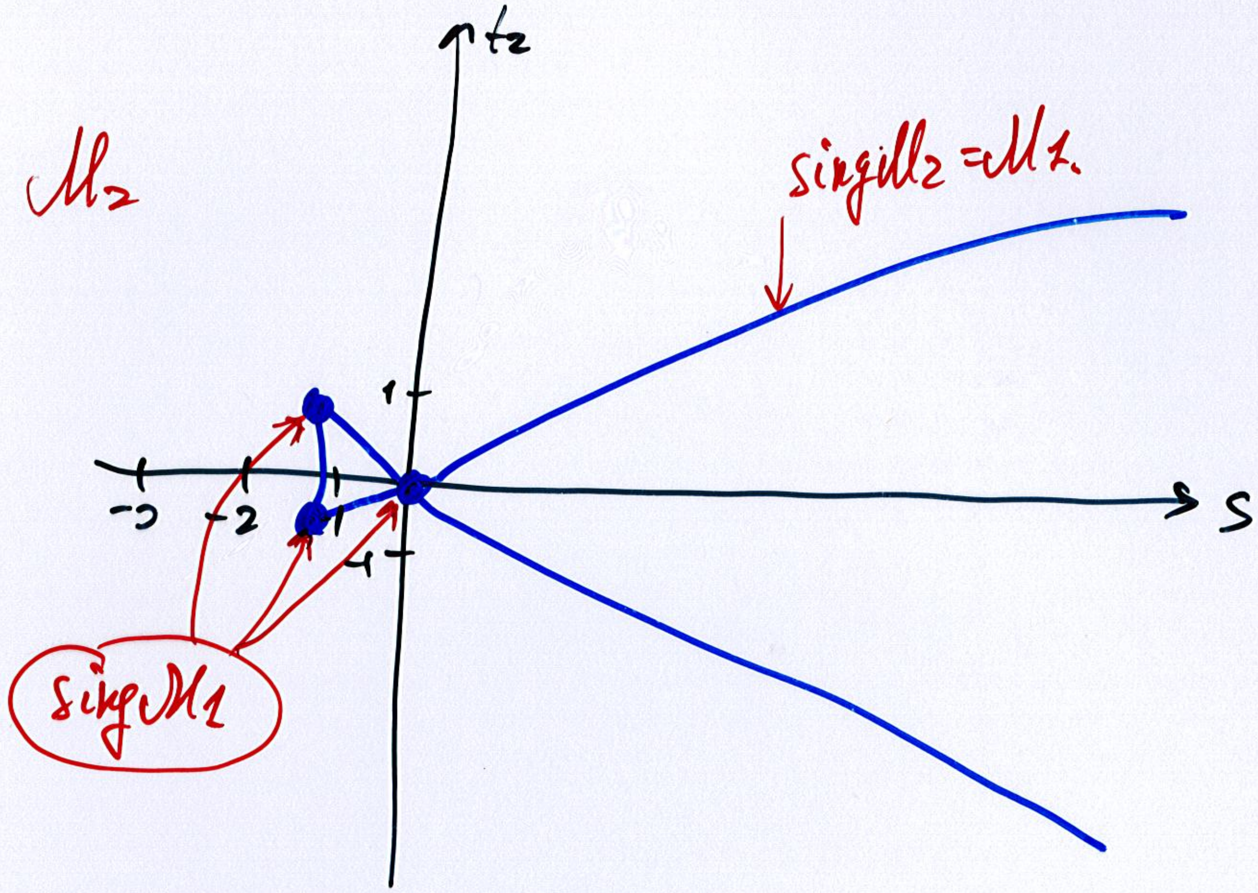
$$\alpha = -u, \quad \delta^2 = \frac{2s}{u} \ (u \neq 0), \quad \delta^2 = 2g \ (u = 0)$$

For sing$\mathcal{M}_1$ $\quad 3t_2 + 4g^2 = 0 \ \Rightarrow$

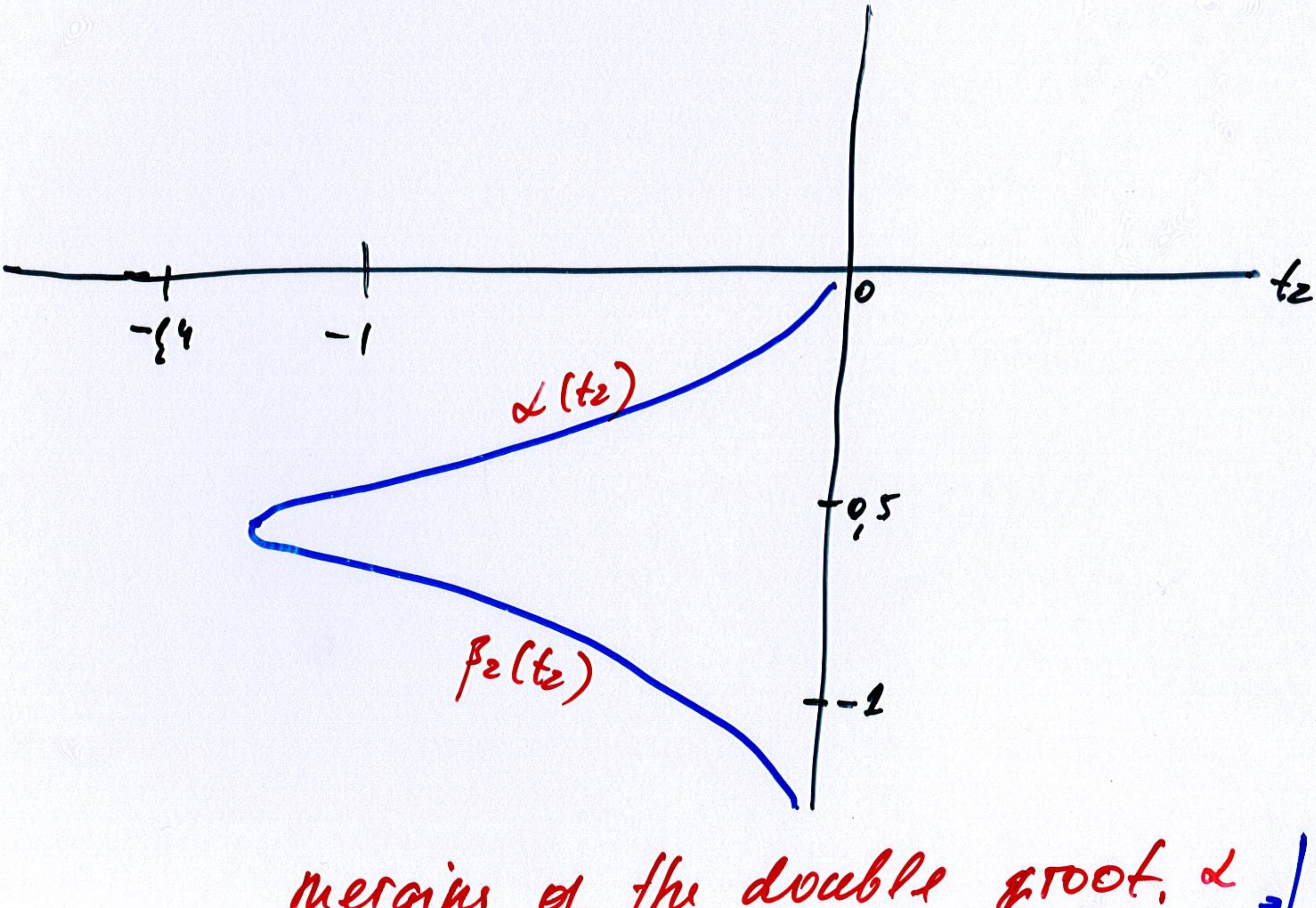
Givental, 1982

$$\text{sing}\mathcal{M}_1 \leftrightarrow \left\{ (s, t_2) = (0, 0), \left(\pm 2\left(\frac{g}{3}\right)^{3/2}, -\frac{4}{3}g^2 \right) \right\}.$$

3_T points



$g = 1.$



merging of the double root α with simple root β_2 → triple root

Branch points $\beta_i(\vec{t})$ as critical points.

Euler-Poisson-Darboux equation.

Introduce the function

$$W(\vec{\beta},\vec{t}) = \oint_{\Gamma_\infty} y_{red}(z,\vec{\beta})\, y(z,\vec{t})\, \frac{dz}{2\pi i}.$$

where

$$y^2_{red}(z,\vec{\beta}) = \prod_{i=1}^{2g}(z-\beta_i) \quad - \quad \text{reduced curve}$$

$$y^2(z,\vec{t}) = W'^2(z) + f(z,t) = P(z,t) \quad \text{full curve}$$

One has as $z \to \infty$

$$y(z,t) = W'(z) + \frac{1}{2}\underbrace{\frac{f(z,\vec{t})}{W'(z)}}_{O\left(\frac{1}{z}\right)} + O\left(\frac{1}{z^{n+2}}\right)$$

!

$\Rightarrow$

$$W(\vec{\beta},\vec{t}) = \oint_{\Gamma_\infty} W'(z)\, y_{red}(z,\vec{\beta}) \frac{dz}{2\pi i} +$$

$$+ \frac{1}{2}\sum_{j=1}^{q} t_j \oint_{\Gamma_\infty} z^{N-j} \frac{y_{red}(z,\vec{\beta})}{W'(z)} \frac{dz}{2\pi i} + \frac{1}{2} t_{q+1}.$$

- linear in $t_1, t_2, \dots, t_q$ $\qquad W(\vec{\beta},\vec{t}) - \frac{1}{2}t_{q+1}$.
- does not depend $t_{q+1}, \dots, t_N$

For cubic model and for $q=1$

$$W(\vec{\beta},\vec{t}) = \frac{1}{128}\Big[-(\beta_1-\beta_2)^2(5\beta_1^2+6\beta_2\beta_1+5\beta_2^2-16g)-32(\beta_1+\beta_2)t_1 + 64 t_2\Big].$$

for $q=2$

$$W(\vec{\beta},\vec{t}) = \frac{1}{256}\Big[2(\beta_1+\beta_2+\beta_3+\beta_4)(\beta_1-\beta_2)^2(\beta_3-\beta_4)^2+\dots\Big]+$$

$$+\dots+\frac{1}{16}\Big[-(\beta_1-\beta_2+\beta_3-\beta_4)^2+4(\beta_1\beta_3+\beta_2\beta_4)+8g\Big]t_1 +$$

$$-\frac{1}{4}(\beta_1+\beta_2+\beta_3+\beta_4)t_2.$$

Properties of $W(\vec{\beta}, \vec{t})$ –

$$(\beta_i - \beta_j) \frac{\partial^2 W}{\partial \beta_i \partial \beta_j} = -\frac{1}{2} \left(\frac{\partial W}{\partial \beta_i} - \frac{\partial W}{\partial \beta_j} \right)$$

$i \neq j = 1, \dots, 2g$

Euler-Poisson-Darboux equation $E(-\frac{1}{2}, -\frac{1}{2})$.

Long story: Darboux book 1894....

Proposition.
For spectral curves in M_g and functions $\vec{\beta} = \vec{\beta}(\vec{t}^{(0)})$ obeying the equations ...
one has

$$\frac{\partial W(\vec{\beta}, \vec{t})}{\partial \beta_i} = 0 \quad \text{at } \vec{\beta} = \vec{\beta}(\vec{t}^{(0)})$$

$i = 1, \dots, 2g$

Proof:
Since $\frac{\partial y_{red}}{\partial \beta_i} = -\frac{1}{2} \frac{y_{red}}{z - \beta_i}$

one has

$$\frac{\partial W}{\partial \beta_i} = -\frac{1}{2} \oint_{\Gamma_\infty} \underbrace{\frac{y_{red}(z, \vec{\beta})\, y(z, \vec{t})}{z - \beta_i}}_{\leftarrow \text{ polynomial}} \frac{dz}{2\pi i}$$

$S_{0,1}$ the branch points $\beta_i(\vec{t})$ are critical points of $W(\vec{\beta}, \vec{t})$!

Equations $\frac{\partial W}{\partial \beta_i} = 0$ is equivalent (for distinct β_i) to the system

$$\oint_{\Gamma_\infty} \frac{z^k}{\sqrt{\prod_{j=1}^{2q}(z-\beta_j)}} \left(W'(z) + \frac{1}{2} \frac{\sum_{j=1}^{q} t_j z^{N-j}}{W'(z)} \right) \frac{dz}{2\pi i} = 0.$$

At $q=1$

$$\oint_{\Gamma_\infty} \frac{W'(z)}{\sqrt{(z-\beta_1)(z-\beta_2)}} dz = 0,$$

$$\oint_{\Gamma_\infty} z \frac{W'(z)}{\sqrt{(z-\beta_1)(z-\beta_2)}} dz = 4\pi i s \qquad s = -\frac{t_1}{4}.$$

- well-known one-cut system in random matrix models.

At $q=2$ two-cuts

$$\oint_{\Gamma_\infty} z^\alpha \frac{W'(z)}{\sqrt{\prod_{j=1}^{4}(z-\beta_j)}} = 0 \qquad \alpha = 0, 1.$$

and

$$\oint_{\Gamma_\infty} z^2 \frac{W'(z)}{\sqrt{\prod_{j=1}^{4}(z-\beta_j)}} dz = 4\pi i S ,$$

$$\oint_{\Gamma_\infty} z^3 \frac{W'(z)}{\sqrt{\prod_{j=1}^{4}(z-\beta_j)}} dz = 4\pi i\left(\frac{1}{2}\sum_{j=1}^{4}\beta_j - N g_N\right)S - i\pi t_2 .$$

new ?!

Distinct β_j , $j=1,..,2q$ - regular sector

- $\mathrm{reg}M_q$

- R_q - set of all solutions

of $\frac{\partial W}{\partial \beta_i} = 0$ with $\beta_i \neq \beta_j$ for all i,j.

~~gap~~ spectral curves in $\mathrm{reg}M_q \leftrightarrow$

$\leftrightarrow$ $(\beta(\bar{t}), \bar{t}) \in R_q$

coalescence of β_j ?

Singular sectors of R_q.

- such $(\vec{\beta}, \vec{t})$ that for the system

$$\frac{\partial W}{\partial \beta_i} = 0 \,, \qquad i = 1, \dots, 2q$$

Jacobian

$$J = \left| \frac{\partial^2 W(\beta, t)}{\partial \beta_i \, \partial \beta_k} \right| = 0$$

Important property of W:

1. at critical point

$$\frac{\partial^2 W}{\partial \beta_i \partial \beta_j} = 0 \qquad i \neq j$$

$\Rightarrow$

2. all derivativs of W can be expressed as combinations of derivatives $\dfrac{\partial^k W}{\partial \beta_i^k}$!

$\Rightarrow$

$$J = \prod_{i=1}^{2q} \frac{\partial^2 W(\vec{\beta}, t)}{\partial \beta_i^2} \,.$$

and

Definition of singular sectors:

Given a set of $2g$ integers $\vec{n} = (n_1, n_2, \dots, n_{2g})$

$n_k \geq 0$,

singular sector $\operatorname{sing} \mathcal{R}_g(\vec{n})$ -
- set of points $(\vec{\beta}, \vec{z}) \in \mathcal{R}_g$ such that

$$\frac{\partial^k W}{\partial \beta_i^{\,k}} = 0 \qquad 1 \leq k \leq n_i + 1,$$

$$\frac{\partial^{n_i+2} W}{\partial \beta_i^{n_i+2}} \neq 0. \qquad i = 1, \dots, 2g.$$

Proposition.
If a curve in $\operatorname{sing} \mathcal{M}_g$ has roots of multiplicities $2n_i + 1$ then $(\vec{\beta}, \vec{z}) \in \operatorname{sing} \mathcal{R}_g(\vec{n})$.

Proof:

$$\frac{\partial^{k+1} W}{\partial \beta_i^{k+1}} \sim \oint_{\Gamma_\infty} \frac{y_{red}(z, \vec{\beta})\, y(z, t)}{(z - \beta_i)^{k+1}} \frac{dz}{2\pi i} =$$

$$= \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial z^{k-1}} \left(\prod_{l=1}^{p} (z - \alpha_l) \prod_{j \neq i} (z - \beta_j) \right) \Bigg|_{z = \beta_i}.$$

Critical behaviour near branch points

Simplification due to the EPD equation.

Let the roots β_{i0} have multiplicities $2n_i+1$ at $\vec{t}_0^{(g)}$. So,

$$\frac{\partial^k W}{\partial \beta_i^k} = 0, \quad 1 \le k \le n_i+1, \qquad \frac{\partial^{n_i+2} W}{\partial \beta_i^{n_i+2}} \neq 0$$

but also

$$! \qquad \frac{\partial^{l_1+\dots+l_{2g}} W}{\partial \beta_1^{l_1} \dots \partial \beta_{2g}^{l_{2g}}} = 0, \qquad 1 \le l_i \le n_i+1.$$

Behaviour of $\vec{\beta}(\vec{t}^g)$ near $\vec{t}_0^g$ - standard technique of multiple scaling limit:

$$t_k = t_{0k} + \varepsilon^{\alpha} t_k^*,$$

$$\beta_i = \beta_{i0} + \varepsilon^{\gamma_i} \beta_i^*$$

$\varepsilon \ll 1$, α and γ_i - to determine!

Keeping the leading terms:

$$\overline{W}(\vec{\beta},\vec{t}) = \overline{W}_0 + \varepsilon^{\alpha} \sum_{k=1}^{q} t_k^{*} A_k +$$

$$+ \varepsilon^{\alpha} \sum_{k=1}^{q} \sum_{l=1}^{2q} \varepsilon^{\gamma_l} t_k^{*} A_{kl} \beta_l^{*} +$$

$$+ \varepsilon^{\alpha} \sum_{k=1}^{q} \sum_{l,m=1}^{2q} \varepsilon^{\gamma_l+\gamma_m} t_k^{*} A_{klm} \beta_l \beta_m^{*} + \sum_{k=1}^{2q} \varepsilon^{(n_k+2)\gamma_k} B_k (\beta_k^{*})^{n_k+2} + \ldots$$

$\Rightarrow$

$$\frac{\partial \overline{W}}{\partial \beta_l^{*}} = \varepsilon^{\alpha+\gamma_l} \sum_{k=1}^{q} t_k^{*} A_{kl} + \varepsilon^{\alpha+\gamma_l} 2 \sum_{k=1}^{q} \sum_{m=1}^{2q} \varepsilon^{\gamma_m} t_k^{*} A_{klm} \beta_m^{*}$$

$$+ \varepsilon^{(n_l+2)\gamma_l} (n_l+2) B_l (\beta_l^{*})^{n_l+1} + \ldots$$

generic case (at least one $A_{kl} \neq 0$)
- second term is subdominant w.r.t. first

$\Rightarrow$ balance of degrees

$$\alpha + \gamma_l = (n_l+2)\gamma_l \quad \Rightarrow$$

$$\gamma_l = \frac{\alpha}{n_l+1} \qquad l = 1, \ldots, 2q$$

So, in generic case (putting $\alpha = 1$)

$$W(\vec{\beta}, \vec{t}) = W_0 + \varepsilon \sum_{k=1}^{g} t_k A_k + \sum_{l=1}^{2g} \varepsilon^{\frac{n_l+2}{n_l+1}} \left(\xi_l \beta_l^{\,k} + B_l (\beta_l^{\,x})^{n_l+2} \right) + \ldots$$

where $\xi_l = \sum_{k=1}^{g} t_k A_{kl}$.

$\Rightarrow$ for $\vec{t}$ and $\vec{\beta}$ near $\vec{t}_0, \vec{\beta}_0$

$$\beta_l - \beta_{l0} \sim \left(\sum_{k=1}^{g} (t_k - t_{0k}) A_{kl} \right)^{\frac{1}{n_l+1}}$$

$l = 1, \ldots, 2g$

So

$$\frac{\partial \beta_l}{\partial t_k} \sim \left(\sum_{k=1}^{g} (t_k - t_{0k}) A_{kl} \right)^{-\frac{n_l}{n_l+1}}$$

as $\vec{t} \to \vec{t}_0$.

Unbounded increase of $\frac{\partial \beta_l}{\partial t_k}$ as $\vec{t} \to \vec{t}_0$

- origin of the break of analyticity of spectral curve and free energy!

Gradient catastrophe for Whitham equations for branch points!

Regularization of gradient catastrophe.

Different mechanisms of regularization

1. Physics - dissipation or dispersion in hydrodynamics, ...

2. Matrix models - quantum corrections or higher genus expansion

3. Integrable systems - use of exact models.. e.g. bi-hamiltonian structure. (Dubrovin 2008..).

....

Direct approach. (naive?!)

To deform the "critical points scheme" by a simple and natural inclusion of derivatives of β_i to prevent the blow up of $\frac{\partial \beta_i}{\partial t_k}$.

without any apriori fixed scheme or knowledge!

Thus, the idea is

to substitute the critical points equat.

$$\frac{\partial W}{\partial \beta_i} = 0$$

near t_0 with $W = W_0 + \dots$ by

Euler-Lagrange equations

$$\frac{\delta W^{reg}}{\delta \beta_i} = 0$$

where

$$W^{reg} = W + \underbrace{\text{terms with } \frac{\partial \beta_i}{\partial t_k}}$$

Natural and old idea.

So,

$$W(\vec{\beta}, t) = W_0 + \varepsilon \sum_{e=1}^{2g} \varepsilon^{\frac{1}{n_e+1}} V_e(\beta_e^{*})$$

where

$$V_e(\beta_e^{*}) = \xi_e \beta_e^{*} + B_e (\beta_e^{*})^{n_e+2}.$$

general case - complicated. $W \to W^{regul.}$

Two rather simple cases.

1. All n_ℓ are different

Simplest nontrivial modification of W

$$W^{reg}(\vec{\beta}^*, \vec{\xi}) = W_0 + \varepsilon \sum_{\ell=1}^{2g} \left(\varepsilon^{\frac{1}{n_\ell+1}} V_\ell(\beta_\ell^*) + \varepsilon^{\delta_\ell} C_\ell \left(\frac{\partial \beta_\ell^*}{\partial \xi_\ell} \right)^2 \right)$$

where C_ℓ are constants, δ_ℓ - to find.

Euler-Lagrange equation

$$\frac{\delta W^{reg}}{\delta \beta_i^*} = \frac{\partial W^{reg}}{\partial \beta_i} - \sum_k \frac{\partial}{\partial \xi_k} \left(\frac{\partial W^{reg}}{\partial \left(\frac{\partial \beta_i^*}{\partial \xi_k} \right)} \right) =$$

$$= \varepsilon^{\frac{1}{n_\ell+1}} \left(\xi_\ell + B_\ell (n_\ell+2)(\beta_\ell^*)^{n_\ell+1} \right) -$$

$$- \varepsilon^{\delta_\ell} 2 C_\ell \frac{\partial^2 \beta_\ell^*}{\partial \xi_\ell^2} = 0$$

$$\Rightarrow \quad \delta_\ell = \frac{1}{n_\ell+1}.$$

Thus,

$$W^{reg}(\vec{\beta}^*, \vec{\xi}) = W_0 + \sum_{l=1}^{2q} \varepsilon^{\frac{n_l+2}{n_l+1}} \left(\xi_l \beta_l^* + B_l (\beta_l^*)^{n_l+2} + C_l \left(\frac{\partial \beta_l^*}{\partial \xi_l} \right)^2 \right).$$

and the Euler-Lagrange equations:

$$2 C_l \frac{\partial^2 \beta_l^*}{\partial \xi_l^2} = \xi_l + (n_l+2) B_l (\beta_l^*)^{n_l+1}$$

$l = 1, \dots, 2q.$

If one $n_l = 1$ then the corresponding eq. is

$$\frac{\partial^2 \Omega}{\partial x^2} = 6\Omega^2 + x$$

where $\beta_l^* = \left(\frac{16 C_l}{B_l} \right)^{1/5} \sqrt{\Omega}$, $\xi_l = \left(\frac{8 C_l^2}{B_l} \right)^{1/5} x$.

Painlevé-I equation

known as the regularizing equation in matrix models (....), for dNLS equation (Dubrovin... (2009)).

|| — regularization of analyticity breaking in formation of third-order root in $y^e(?)$.

2. Some n_i coincide, i.e.

$$n_1 = n_2 = \dots = n_k = n$$

In this case

$$\overline{W} = \overline{W}_0 + \varepsilon^{\frac{n+2}{n+1}} \sum_{l=1}^{k} \left(\xi_l \beta_l^* + B_l (\beta_l^*)^{n+2} \right) + \varepsilon \sum_{l=k+1}^{2g} \varepsilon^{\frac{1}{n_l+1}} \left(\xi_l \beta_l^* + B_l (\beta_l^*)^{n_l+2} \right).$$

Contributions of $\beta_1^*, \dots, \beta_k^*$ are of the same order.

So, W^{reg} should naturally contain a mixture of derivatives of $\beta_1^*, \dots, \beta_k^*$.

Thus, the natural form is

$$\overline{W}^{reg} = \overline{W}_0 + \varepsilon^{\frac{n+2}{n+1}} \Big[\sum_{l=1}^{k} \left(\xi_l \beta_l^* + B_l (\beta_l^*)^{n+2} \right) + \frac{1}{2} \sum_{m,p,q=1}^{k} D_{mpq} \beta_m^* \frac{\partial \beta_p^*}{\partial \xi_q} + \frac{1}{2} \sum_{m,p,q,t=1}^{k} D_{mpqt} \frac{\partial \beta_m^*}{\partial \xi_p} \frac{\partial \beta_q^*}{\partial \xi_t} \Big] + \varepsilon \sum_{l=k+1}^{2g} \varepsilon^{\frac{1}{n_l+1}} \left(\xi_l \beta_l^* + B_l (\beta_l^*)^{n_l+2} + C_l \left(\frac{\partial \beta_l^*}{\partial \xi_l} \right)^2 \right).$$

Euler-Lagrange equations for $\beta_1^*, \dots, \beta_k^*$

$$\sum_{m,p,q=1}^{k} D_{impq} \frac{\partial^2 \beta_p^*}{\partial \xi_m \partial \xi_q} - \sum_{m,p=1}^{k} D_{imp} \frac{\partial \beta_m^*}{\partial \xi_p} =$$

$$= \xi_i + (n+2) B_i \left(\beta_i^*\right)^{n+1}$$

$i = 1, \dots, k.$

Simplest cases: $k = 2$, $D_{impq} = 0$

$$\left.\begin{aligned} \frac{\partial \beta_1^*}{\partial \xi} &= D_{122}\xi + C\eta + (n+2) B_2 \left(\beta_2^*\right)^{n+1}, \\ \frac{\partial \beta_2^*}{\partial \xi} &= -D_{121}\xi - A\eta - (n+2) B_1 \left(\beta_1^*\right)^{n+1} \end{aligned}\right\}$$

where $\xi_1 = D_{121}\xi + A\eta$, $\xi_2 = D_{122}\xi + C\eta$

A, C - arbitrary constants

For $n = 1$, $A = C = 0$

it is the special Riccati system

For $n = 2$, $A = C = 0$

– two component extension of Abel equation

The case $\underline{k = 2, \ \mathcal{D}_{mpq} = 0}$

the system restricted to subspace $\xi_1 - \xi_2 = 0$

is

$$\left.\begin{aligned} \frac{\partial^2 \beta_1^k}{\partial \xi^2} &= A_{11} \beta_1^{k n+1} + A_{12} \beta_2^{k n+1} + A_1 \xi, \\ \frac{\partial^2 \beta_2^k}{\partial \xi^2} &= A_{21} \beta_1^{k n+1} + A_{22} \beta_2^{k n+1} + A_2 \xi, \end{aligned}\right\}$$

where $\xi = \frac{1}{2}(\xi_1 + \xi_2)$, and $A_{ik}, A_k \ldots$

At $n = 1$. – two-component extension of Painlevé-I equation ?

Possible extensions

Generic polynomial $P(z,\bar{z})$

$$P(z,\bar{z}) = z^N + \sum_{k=1}^{N} z^{N-k} t_k .$$

General factorization

$$P(z,\bar{z}) = \prod_{i=1}^{r} (z - x_i)^{m_i} \qquad m_i \geq 1.$$

where $\sum_{i=1}^{r} m_i = N.$

$\mathcal{M}_m$ - set of all $P(z,\bar{z})$ of such form with fixed $\overleftrightarrow{m} = (m_1, \dots, m_r)$.

from $\prod_{i=1}^{r} (z - x_i)^{m_i} = z^N + \sum_{k=1}^{N} z^{N-k} t_k$

$N-r$ constraints

$$f_k(t_1, t_2, \dots, t_N) = 0 \quad , \quad k = 1, \dots, N-r$$

Function $W_{m,\kappa,I}(\vec{x},\vec{t})$

$$W_{m,\kappa,I} = \oint_{\Gamma_\infty} \frac{dz}{2\pi i} \prod_{i\in I} (z-x_i)^{(1-\kappa)m_i} \left(P(z,\vec{t})\right)^{\kappa}.$$

where

$$I \subset \{1,2,\dots,r\}$$

and κ is real positive such that

$$0 < (1-\kappa)m_i < 1, \qquad i \in I$$

$$\kappa m_i \text{ - integer}, \quad i \notin I$$

EPD system

$$(x_i - x_j)\frac{\partial^2 W_{m,\kappa,I}}{\partial x_i \partial x_j} = (\kappa-1)\left(m_j \frac{\partial W_{m,\kappa,I}}{\partial x_i} - m_i \frac{\partial W_{m,\kappa,I}}{\partial x_j}\right)$$

Proposition.

$$\frac{\partial W_{m,\kappa,I}}{\partial x_i} = 0 \iff x_i = x_i(\vec{t}).$$