Spectral curves in gauge/string duality.
Integrability, singular sectors and regularization.

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Поздравляю, Миша!

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Cachazo, Dijkgraaf, Intriligator, Vafa
spectral curves

Class of hyperelliptic curves

\[ y^2 = W'(z)^2 + f(z) \]

\( W(z) \) and \( f(z) \) - polynomials
\[ \text{deg } f = \text{deg } W - 2 \]

- description of low energy dynamics of \( N=2 \) \( U(n) \) su.sy. gauge theories via duality between su.sy. gauge theory and string theories on deformed 3-dim Calabi-Yau manifolds defined by

\[ W'(z)^2 + f(z) + u^2 + v^2 + w^2 = 0 \]

process of Geometric Transition
Other appearances:

1. Subclass of Seiberg-Witten curve

\[ y^2 = W'(z)^2 + \Phi(z) \]

\[ \deg W' = N_f, \quad \deg \Phi = N_f, \quad N_f \leq 2N_c \]

COIV curves \( N_f = N_c - 1 \).

2. Asymptotic eigenvalue distribution for matrix models with

\[ Z_n = \int e^{-n \text{tr}(W(M))} \, dM \]

as \( n \to \infty \).

3. Asymptotic distribution of zeros for orthogonal pol. \( P_n(z) = z P_{n-1} \ldots \) with

\[ \int P_n(z) \, e^{-n W(z)} \,dz = 0, \quad k=0,1,\ldots \]

as \( n \to \infty \)

\[ \cdots \]
CDIV curves have branch points

\[ 2 \mathfrak{g} \ (1 \leq \mathfrak{g} \leq \mathfrak{n}) \]

- odd-order roots \( \vec{\beta} = (\beta_1, \ldots, \beta_{2\mathfrak{g}}) \)
  of pol. \( y^2(\tau) \).

9 cuts \( \gamma_i = [\beta_{2i-1}, \beta_{2i}] \quad i = 3 \ldots 9 \).

at \( f(\tau) = 0 \) cuts shrink into points (double roots) \( \mathcal{W}(\mathfrak{g}) \).

Important characteristics:

't Hooft parameters \( \mathcal{F} = (S_1, \ldots, S_{2\mathfrak{g}}) \)
defined by (period relations)

\[
\oint_{A_j} y(\tau) d\tau = -4\pi i S_j, \quad j = 3 \ldots 9
\]

where \( A_j \) - counterclockwise cycles around \( S_j \).

and \( f(\tau) = \mathcal{W}(\mathfrak{g}) + O(1) \) as \( \tau \to \infty \)

Free energy

\[
\mathcal{F} = \oint s(\tau) \mathcal{W}(\mathfrak{g}) d\tau + \frac{1}{2} \oint \int s(\tau) g(\tau) \delta(\tau - \tau') d\tau d\tau'
\]

density \( s(\tau) \)

\[
s(\tau) = \frac{y(\tau^+)}{2\pi i} = -\frac{y(\tau^-)}{2\pi i}.
\]
Study of moduli space for CDIV curves.

\[ W(z) = \frac{z^{N+1}}{N+1} + \sum_{n=1}^{N} g_n z^n, \]

\[ f(t) = \sum_{k=1}^{N} t_k \bar{z}^{N-k} \]

ge - fixed, \( \bar{z} = (t_1, \ldots, t_N) \) - deformation parameters.

Study of analytic properties of roots for \( f(t) \)

\[ \Rightarrow \text{analytical properties of density } \rho \text{ and free energy } \mathcal{F} \]

as the functions of \( S_i \) and \( t_k \) !

In general

\[ g^2(t) = \prod_{e=1}^{\rho} (z - t_e)^{2\rho} \prod_{i=1}^{2g} (z - \beta_i)^{2g_i + 1} \]

with \( N = \frac{\rho}{2} + \sum_{i=1}^{2g} \frac{2g_i}{2} + 1 \)

\[ p + 2q - N \text{ independent parameters} \]

\[ p \leq 1 \]

\[ q_i \geq 0. \]
Subclass of CDIV curves completely defined by q t'Kooft parabola

\[ y^2(z) = \prod_{\ell=1}^{P} (z - d_\ell)^2 \prod_{i=1}^{2q} (z - \beta_i) \]

\[ p+q = N. \]

**Definition.**

\( \mathcal{M}_q \) - set of all CDIV curves which have 2q branch points (q cuts) at most.

\( q = 0, 1, \ldots, N \) - given.

\( M_{\text{regq}} \cdots C_{\text{Mq}}. \)

**Definition.**

Regular sector \( \text{regq} \) - subset of all curves in \( \mathcal{M}_q \) with all roots \( (2, \ell, \beta) \) distinct.

Singular sector

\( \text{sing}\ M_q = M_q / \text{regq}. \)
Whitham equations

Reduced curve
\[ y^2_{\text{red}} = \prod_{i}^{2g} (z - \beta_i) \]

One has
\[ \left( \frac{W'(z)}{y_{\text{red}}(z)} \right)_{\theta} = \prod_{e=1}^{g} (z - \beta_e) \]

\[ \Rightarrow \quad \theta e = \theta e(\beta_1, \ldots, \beta_{2g}) \]

Equations for \( \beta_e \):

Standard approach.

I. Krichever (1994),

\[ \ldots \]

L. Chekhov et al. (2003)...

\[ \ldots \]

Riemann surfaces Abelian differential function \( \psi(z) \), \( \ldots \)
Noromorphic differential

$$dS = \frac{1}{2} (y(z) + W(z)) dz$$

on $N_0$.

where

$$y(z) = W(z) - \frac{2s}{z} + O(z^{-2})$$

$$s = \frac{i}{2} S_i$$

One can show that

$$dS = \sum_{i=1}^{N+1} d\lambda_i - \sum_{j=1}^q S_j (d\lambda_0 + 2\pi i (1-\delta_{ij}) d\Psi_j)$$

where

$$d\Psi_j = \frac{R_i(z)}{Y}(z) dz, \quad d\lambda_i = \left( \frac{1}{2} \frac{z^i \cdot R_i(z)}{Y(z)} \right) dz$$

- Abelian diff. of first second and third kind

$$\Rightarrow \frac{\partial dS}{\partial S_j} = -d\lambda_0 - 2\pi i (1-\delta_{ij}) d\Psi_j. \quad \Rightarrow$$

$$\frac{\partial y(z)}{\partial S_j} = -2 \frac{P_i(z) - 2\pi i (1-\delta_{ij}) A \cdot (\beta \vec{p})}{Y(z)}$$

$$\Rightarrow$$

$$\frac{\partial B_i}{\partial S_j} = 4 \frac{P_i(\beta_i, \vec{p}) - 2\pi i (1-\delta_{ij}) A \cdot (\beta \vec{p})}{\prod_{\ell=1}^{\ell_0} (\beta_i - \beta_\ell) \prod_{k \neq i} (\beta_i - \beta_k)}$$

prepotential, $\Sigma$-function
CDIV curves in terms of deformation parameters.

Curves in $\mathcal{U}$ satisfy 2N equation:

\[
P(\mathcal{Z}(t), t) = p'(\mathcal{Z}(t), t) = 0, \quad i = 1, \ldots, 2N.
\]

\[
P(\mathcal{Z}(t), t) = 0, \quad i = 1, \ldots, 2N.
\]

where

\[
P(\mathcal{Z}(t)) = \sum_{k=1}^{N} t_k Z^{n_k}.
\]

Jacobian of the system

\[
\frac{\delta}{\partial (\cdot)}
\]

- non-singular minors
- from implicit function theorem:

Theorem. In regular each set of 9 parameters $(a_i, b_i, c_i)$ is an analytic system of coordinates, i.e. all the variables $(\mathcal{Z}, \mathcal{Y}_i, t_k)$ are local analytic functions of $(t_1, \ldots, t_2)$. 
Eliminating $(2, \beta, i)$ from above system one gets $N - 9$ constraints

$$f_k(T) = 0, \quad k = 1, \ldots, N - 9$$

which characterized $M_9$ in the space $C^n$ of deformation parameters.

Class $\mathcal{M}_N$ contains all C-DIV curves.

In region $\mathcal{M}_N$

$$\frac{\partial S_i}{\partial \kappa_k} = -\frac{1}{8\pi i} \oint \frac{z^{n-k}}{A_i \sqrt{P(z-\beta)}} \, dz \quad i, \, \kappa = 2, \ldots, N$$

→ Jacobian of trans: $(t_2, \ldots, t_N) \rightarrow (S_2, \ldots, S_N)$ is non singular (it is the matrix of $A$-periods for abelian differential)

→ Change of variable $(T) \rightarrow (S)$ is a pure analytic diffeomorphism locally in region $M_9$. 
Whitham equations for branch points w.r.t. deformation parameters.

\[ \beta_i = \beta_i(t_1, \ldots, t_n) \]

From \[ P(z, t) = \gamma^2(z, t) = W(z) + \sum_{k=1}^{2} \sum_{\mu=0}^{N-\mu} \frac{1}{k!} (z - z_{\mu})^k \]

one gets

\[ t_{e+q} = \sum_{N=1}^{P} Vem(\tilde{z}) \left[ P(d\mu, \tilde{z}) - (W'(d\mu)) \sum_{j=1}^{9} f_j \cdot d\mu_j \right] \]

\[ e = 1, \ldots, P = N - 9 \]

and \[ Vem(\tilde{z}) - \text{inverse of Vandermonde matrix } (x^{P-m})_e \epsilon_{m=1} \]

\[ \Rightarrow P(\beta_i; \tilde{z}) - \sum_{e, \mu=1}^{P} \beta_i \epsilon_{e, \mu=1} Vem(\tilde{z}) P(d\mu, \tilde{z}) \]

do not depend on \((t_{e+1}, \ldots, t_n)\).
Differentiating above equations w.r.t. $t_1, \ldots, t_q$ and taking into account $P(\beta, \theta) = P'(\beta, \theta) = 0$, one gets system

$$\frac{\partial \beta_i}{\partial t_j} = \frac{\sum_{k \neq i} \beta_k^{N-i} \cdot \text{Vom}(i)x_k^{N-i} - \beta_i^{N-i}}{\prod_{c \neq i} (\beta_i - \beta_c)^2 \prod_{j \neq i} (\beta_i - \beta_j)}$$

$c = 2, \ldots, q$,

$j = 2, \ldots, q$.

Examples:

$q = N$

$$\frac{\partial \beta_i}{\partial t_j} = -\frac{\beta_i^{N-j}}{\prod_{j \neq i} (\beta_i - \beta_j)}.$$ 

$q = N-1$

$$\frac{\partial \beta_i}{\partial t_j} = \frac{x^{N-j} - \beta_i^{N-j}}{(x - \beta_i)^2 \prod_{j \neq i} (\beta_i - \beta_j)},$$

$$x = -Ng - \frac{\beta_1 + \ldots + \beta_q}{2}.$$
Cubic model \((N = 2)\)

\[
W(z) = \frac{z^3}{3} - g \cdot z,
\]

\(g \in \mathbb{C} \setminus \{0\}\)

and

\[
y^2(z, t) = (z^2 - g)^2 + t_1 z + t_2,
\]

\(t_1 = -4g\)

Generic class \(\mathcal{C}Z\)

\[
y^2(z, t) = \prod_{i=1}^{\mathcal{C}Z} (z - \rho_i)
\]

In terms of variables \(U_1, U_2, \delta_1, \delta_2\) defined as

\[
\beta_1 = U_1 + \delta_1, \quad \beta_2 = U_1 - \delta_1, \quad \beta_3 = U_2 + \delta_2, \quad \beta_4 = U_2 - \delta_2.
\]

One key equation

\[
y U_1^6 - U_1^4 U_2^2 - t_2 U_1^2 - \delta_2 = 0
\]

and

\[
U_2 = -U_1, \quad \delta_1 = g - U_1^2 + \frac{\delta}{U_1}, \quad \delta_2 = g - U_1^2 - \frac{\delta}{U_1},
\]

that defines \(\beta_i(z, t_2) \quad i = 1, 2, 3, 4\).
Regular sector regullz
\((s,t) \in C^2\) with
\[2T^4 + (18gtz + 16g^2)S^2 - t_2^2 - g^2t_2^2 \neq 0.\]

\(\text{Sing} M_z = M_z\)
for \(M_z\).
\[y^2(z^*) = (z - \alpha)(z - \beta_1)(z - \beta_2)\]
In terms of \(u, \delta, \beta_1 = u + \delta, \beta_2 = u - \delta\).
one has
\[u^2 - gu + S = 0\]
and
\[z = -u, \quad \delta = \frac{2S}{u}(u + 0), \quad \delta^2 = 2g \quad (u = 0)\]

For \(\text{sing} M_z\)
\[T^2 + g^2 = 0 = 6i\text{ental, } 1982\]
\(\text{Sing} M_z \leftrightarrow \left\{ (s,t) = (0,0), (\pm 2(\frac{3}{4})^{\frac{7}{2}}, -\frac{4}{3} g^2) \right\}\)
37 points
\[ M_2 \]

\[ \text{Singul}_2 = M_2. \]

\[ g = 2. \]

\[ \text{merging of the double root, } \lambda(\xi_2) \text{ with simple root } \beta_2. \]
Branch points $p_i(z)$ as critical points.

Euler-Poisson-Darboux equation.

Introduce the function

$$W(x, t) = \oint_{\Gamma_0} y_{\text{red}}(z, p) \, y(z, t) \frac{dz}{2\pi i}$$

where

$$y_{\text{red}}(z, p) = \prod_{i=1}^{2g} (z - p_i)$$

reduced curve

$$y^2(z, t) = W'(z) + f(z, t) = \nu(z, t)$$

full curve

One has as $z \to \infty$

$$y(z, t) = W'(z) + \frac{1}{\pi^2} \frac{f(z, t)}{W'(z)} + O\left(\frac{1}{z^{x+2}}\right)$$

$$O(\frac{1}{z})$$
\[
W(\beta, \xi) = \frac{1}{\pi} \int \frac{W'(z) Y_{\text{red}}(z, \bar{\beta})}{W(z)} \frac{d^2 \xi}{2\pi i} + \\
+ \frac{1}{2} \sum_{j=1}^{g} t_j \int_{\Gamma_0} \frac{Z^{-N-j} Y_{\text{red}}(z, \bar{\beta})}{W(z)} \frac{d^2 \xi}{2\pi i} + \frac{1}{2} t_{g+1}.
\]

- linear in \( t_2, t_3, \ldots, t_g \)
- does not depend on \( t_{g+1}, \ldots, t_N \)

For cubic model and for \( q = 1 \)
\[
W(\beta, \xi) = \frac{1}{128} \left[ -(\beta - \beta_2)^2 (5\beta_1 + 6\beta_2 + 5\beta_2^2 - 16 \xi) - 32(\beta_1 + \beta_2) \xi + 64 \xi^2 \right].
\]

For \( q = 2 \)
\[
W(\beta, \xi) = \frac{1}{256} \left[ 2(\beta_1 + \beta_2 + \beta_3 + \beta_4)(\beta_1 - \beta_2)^2 (\beta_3 - \beta_4)^2 \right] + \\
+ \frac{1}{16} \left[ -(\beta_1 - \beta_2 + \beta_2 - \beta_3)^2 + 4(\beta_1 \beta_3 + \beta_2 \beta_4) + 8g \right] t_3 + \\
- \frac{1}{4} (\beta_1 + \beta_2 + \beta_3 + \beta_4) t_2.
\]
Properties of \( W(\beta, \xi) \) -

\[
(\beta_i - \beta_j) \frac{\partial^2 W}{\partial \xi_i \partial \xi_j} = -\frac{1}{2} \left( \frac{\partial W}{\partial \xi_i} - \frac{\partial W}{\partial \xi_j} \right)
\]

Equation \( E(-\frac{1}{2}, -\frac{1}{2}) \)

Long story: Darboux 1894...

Proposition.

For spectral curves in \( MG \) and functions \( \beta = \beta(x, y) \) obeying the equations...

one has

\[
\frac{\partial W(\beta, \xi)}{\partial \xi_i} = 0 \text{ at } \beta = \beta(x, y)
\]

Proof: Since

\[
\frac{\partial Y_{red}}{\partial \xi_i} = -\frac{1}{2} \frac{Y_{red}}{z - \beta_i}
\]

one key

\[
\frac{\partial W}{\partial \xi_i} = -\frac{1}{2} \int_{\mathbb{R}^2} \frac{Y_{red}(z, \beta) Y(z, \xi)}{z - \beta_i} \, dz
\]

\( \sim \) polynomial
So, the branch points $\beta_i'(x)$ are critical points of $W(\beta, z)$.

Equations $\frac{2W}{\partial \beta_i} = 0$ is equivalent (for distinct $\beta_i$) to the system

$$
\oint_{\Gamma_0} \frac{z^k}{\sqrt{(z-\beta)(z-\beta^*)}} \left( W'(z) + \frac{1}{2} \sum_{j=1}^{2g} \gamma_j(z, z_0) \right) dz = 0.
$$

At $g=1$

$$
\oint_{\Gamma_0} \frac{W'(z)}{\sqrt{(z-\beta)(z-\beta^*)}} dz = 0.
$$

$$
\oint_{\Gamma_0} \frac{z W'(z)}{\sqrt{(z-\beta)(z-\beta^*)}} dz = 4\pi \gamma_1 \gamma_2 \gamma_3 \gamma_4.
$$

- well-known one-cut system in random matrix models.

At $g=2$, two cuts

$$
\oint_{\Gamma_0} z^2 \frac{W'(z)}{\sqrt{\prod_{i=1}^{4} (z-\beta_i)}} = 0, \quad z = 0, 2.
$$
\[
\int_\mathcal{D} z^2 \frac{W'(z)}{\sqrt{\prod_{j=1}^l (z - \beta_j)}} dz = 4\pi i \delta_S \\
\int_{\mathcal{R}^q} z^3 \frac{W'(z)}{\sqrt{\prod_{j=1}^l (z - \beta_j)}} dz = 4\pi i (\frac{1}{2} \sum_{j=1}^l \beta_j - N) S - \text{int}_2
\]

new?!

Distinct \( \beta_j, j = 1, \ldots, 29 \) regular sector

- regularly

- \( \mathcal{R}^q \) set of all solutions

of \( \frac{\partial W}{\partial \beta_i} = 0 \) with \( \beta_i \neq \beta_j \) for all \( i \neq j \).

spectral curves in regularly \( \leftrightarrow \)

\( (\beta(\mathbb{C}), \mathcal{I}) \in \mathcal{R}^q \)

coalescence of \( \beta_i \)?
Singular sectors of $R_q$.

- such $(\bar{\beta}, \bar{\tau})$ that for the system

\[
\frac{\partial W}{\partial \beta_i} = 0, \quad i = 1 \ldots, 29
\]

Jacobian

\[
y = \left| \frac{\partial^2 W(\beta, \tau)}{\partial \beta_i \partial \beta_j} \right| = 0
\]

Important property of $W$:

1. at critical point

\[
\frac{\partial^2 W}{\partial \beta_i \partial \beta_j} = 0, \quad i \neq j
\]

\Rightarrow

2. all derivatives of $W$ can be expressed as combinations of derivatives $\frac{\partial^2 W}{\partial \beta_i^2}$.

\[
y = \prod_{i=1}^{29} \frac{\partial^2 W(\beta, \tau)}{\partial \beta_i^2}
\]

and
Definition of singular sectors:

Given a set of 2g integers \( \vec{\nu} = (n_1, n_2, \ldots, n_{2g}) \) with \( n_k \geq 0 \), the singular sector \( \text{sing } R_g(\vec{\nu}) \) is the set of points \( (\vec{\beta}, \vec{\tau}) \in R_g \) such that

\[
\frac{\partial^k W}{\partial \beta_i^{-k}} = 0, \quad 1 \leq k \leq n_i + 1, \\
\frac{\partial^{n_i+2} W}{\partial \beta_i^{n_i+2}} \neq 0, \quad i = 1, \ldots, 2g.
\]

Proposition:

If a curve in \( \text{sing } R_g(\vec{\nu}) \) has roots of multiplicity \( 2\nu_i + 1 \) then \( (\vec{\beta}, \vec{\tau}) \in \text{sing } R_g(\vec{\nu}) \).

Proof:

\[
\frac{\partial^k W}{\partial \beta_i^{-k}} = \oint \frac{\text{Res} (\vec{\beta}, \vec{\nu}) y(\vec{\tau} + \vec{z})}{(\vec{\tau} - \beta_i)^{k+1}} \frac{d\vec{z}}{2\pi i} = \\
= \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial \beta_i^{-k-1}} \left( \prod_{e=1}^{g} \left[ \prod_{j=1}^{n_e} \left( \tau_j - \beta_i \right) \right] \right) \bigg|_{\vec{\tau} = \vec{\beta}_i}.
\]
Simplification due to the E.P.D equation.

Let the roots $\rho_i$ have multiplicity $2\nu + 1$ at $T_0$. So,

$$\frac{\partial^2 W}{\partial \beta_i^2} = 0 \quad (1 \leq \nu \leq \nu + 1)$$

But also,

$$\frac{\partial^{\nu + 1} W}{\partial \beta_1 \partial \beta_2 \cdots \partial \beta_{\nu + 1}} = 0 \quad (1 \leq \nu \leq \nu + 1)$$

Behaviour of $\tilde{\beta}(T)$ near $T_0$ - standard technique of multiple scaling limit:

$$t_k = t_0 + \varepsilon^{2\nu} t_k^*$$

$$\beta_i = \beta_{i0} + \varepsilon \gamma_i \beta_i^*$$

$\varepsilon \ll 1$, $d$ and $\gamma_i$ - to determine!
Keeping the leading terms,

\[
\overline{W}(\beta, \xi) = W_0 + \varepsilon^{d+\xi} \sum_{k=1}^{q} t_k \xi^k A_k + \\
+ \varepsilon^{d+\xi} \sum_{k=1}^{q} \sum_{e=1}^{2q} \varepsilon e t_k \xi^k A_{ke} \beta_e^k + \\
+ \varepsilon^{d+\xi} \sum_{k=1}^{q} \sum_{e=1}^{2q} \sum_{\mu=1}^{2q} \varepsilon (\mu+2) t_k \xi^k A_{ke} \beta_e^k \beta_{ke}^\mu + \sum_{k=1}^{q} \varepsilon (\mu+2) t_k \xi^k B_k (\beta_k)^{\mu+2} + ...
\]

\[
\frac{\partial \overline{W}}{\partial \beta_e^k} = \varepsilon^{d+\xi} \sum_{k=1}^{q} t_k \xi^k A_{ke} + \varepsilon^{d+\xi} \sum_{k=1}^{q} \sum_{\mu=1}^{2q} \varepsilon \xi^k \xi^\mu A_{ke} \beta_e^\mu
\]

\[
+ \varepsilon (\mu+2) \xi^e (\mu+2) B_k (\beta_k)^{\mu+1} + ...
\]

**Generic case (at least one \( A_{ke} \neq 0 \))**
- Second term is subdominant w.r.t. first

\[
\Rightarrow \quad \text{balance of degrees}
\]

\[
x + \xi^e = (\mu+2) \xi^e \Rightarrow \xi^e = \frac{x}{\mu+1}
\]

\[
\xi^e = \frac{x}{\mu+1}
\]

\[
e = 1, \ldots, 2q
\]
So, in generic case

\[ W(\alpha, \varepsilon) = W_0 + \varepsilon \sum_{k=1}^{9} E_k A_k + \sum_{l=1}^{29} E_l \frac{A_l^{e+2}}{E_l e+2} \left( E_l e B_e + B_e (B_e^{e+2}) E_l e \right) \]

where \( E_r = \sum_{k=1}^{9} E_k A_k \).

\[ \beta_e - \beta_e^0 \approx \left( \sum_{k=1}^{9} (E_k - t_0) A_k \right) \frac{1}{E_l e+1} \]

\[ \frac{\partial \beta_e}{\partial t_k} \approx \left( \sum_{k=1}^{9} (E_k - t_0) A_k \right) \frac{-1}{E_l e+1} \]

as \( \varepsilon \to 0 \).

Unbounded increase of \( \frac{\partial \beta_e}{\partial t_k} \) as \( \varepsilon \to 0 \).

Origin of the break of analyticity

of spectral curve and free energy.

Gradient catastrophe for Whitham equations for branch points!
Regularization of gradient catastrophe.

Different mechanisms of regularization

1. Physics - dissipation or dispersion in hydrodynamics, ....

2. Matrix models - quantum corrections or higher genus expansion

3. Integrable systems - use of exact models...
   e.g. bi-Hamiltonian structure. (Dubrovin 2008...)

Direct approach. (naive?!) To deform the "critical points scheme" by a simple and natural inclusion of derivatives of $g_i$ to prevent the blow up of $\frac{\partial}{\partial x}$ without any apriori fixed scheme or knowledge!
Thus, the idea is to substitute the critical points equal:

\[ \frac{\partial W}{\partial \beta_i} = 0 \]

near \( \theta_0 \) with \( W = W_0 + \ldots \) by Euler-Lagrange equations

\[ \frac{\delta W}{\delta \beta_i} = 0 \]

where

\[ W_{\text{reg}} = W + \text{terms with } \frac{\partial W}{\partial \beta_k} \]

Natural and old idea.

So,

\[ W(\beta, t) = W_0 + \varepsilon \sum_{e=1}^{2g} \sum_{k=1}^{s_{e,k+1}} U_e(\beta^e) \]

where

\[ U_e(\beta^e) = \xi e^{\beta^e} + Be(\beta^e)^{m+2} \]

general case - complicated: \( W \rightarrow W^{\text{reg}} \).
Two rather simple cases.

1. All we are different

Simplest nontrivial modification of $W$

$$W_{\text{reg}}(\beta, \xi) = W_0 + \varepsilon \sum_{e=1}^{e_2} \left( \frac{1}{3 \text{He}_{\text{reg}}(\beta_e)} + \varepsilon C_e \left( \frac{\partial \beta_e}{\partial \xi_e} \right)^2 \right)$$

where $C_e$ are constants, $\delta e - do$ find.

Euler–Lagrange equation

$$\frac{\delta W_{\text{reg}}}{\delta \beta_e} = \frac{\partial W_{\text{reg}}}{\partial \beta_e} - \sum_{e} \frac{\partial}{\partial \xi_e} \left( \frac{\partial W_{\text{reg}}}{\partial (\partial \beta_e / \xi_e)} \right) =$$

$$= \frac{1}{\text{He}_{\text{reg}}} \left( \xi_e + \varepsilon e(\xi_e^2)(\beta_e)^{\nu e+1} \right) -$$

$$- \varepsilon \delta e \left( 2 C_e \frac{\beta_e}{\eta \xi_e^2} \right) = 0$$

$$\Rightarrow \quad \delta e = \frac{1}{\text{He} + 1}.$$
Thus,

\[ W^{\text{ret}}(\beta^*, \xi) = W_0 + \sum_{\ell=1}^{2q} E_\ell \left( \beta^* \right) \left( \xi_\ell \beta^* + B(\beta^*) \right) \frac{\partial E_\ell}{\partial \xi_\ell} + C_e \left( \frac{\partial E_\ell}{\partial \xi_\ell} \right)^2. \]

are the Euler-Lagrange equations

\[ 2C_e \frac{\partial^2 \beta^*}{\partial \xi_\ell^2} = \frac{\partial E_\ell}{\partial \xi_\ell} + (\ell e + 2) B_e (\beta^*)^{\ell e + 1} \]

\[ \ell = 1, \ldots, 2q. \]

If one \( \ell e = 1 \) then the corresponding eq. \( \ell \)

\[ \frac{\partial^2 \mathcal{L}}{\partial x^2} = 6 \mathcal{L}^2 + x \]

where \( \beta^* = \left( \frac{16C_e}{\beta_e} \right)^{\frac{1}{2}}, \quad \xi_e = \left( \frac{8C_e^2}{\beta_e} \right)^{\frac{1}{2}} x. \]

**Painlevé-I equation**

Known as the regularization equation in matrix models\(^{(\ldots)}\), for dNLS equation (Dubrovin\(, (2003))\).

Regularization of analyticity breaking in formation of third-order root in \( y^3 \).
2. Some $n_i$ coincide, i.e.

$$n_L = n_e = \ldots = n_R = n$$

In this case

$$W = W_0 + \sum_{e=1}^k \left( \xi_e \beta_e^k + B_e (\beta_e^e) \right) +$$

$$+ \sum_{e=k+1}^{2q} \frac{1}{\xi_e^{k+1}} \left( \xi_e \beta_e^e + B_e (\beta_e^e) \right)$$

Contributions of $\beta_1^e, \ldots, \beta_k^e$ are of the same order.

So, $W_{\text{reg}}$ should naturally contains a mixture of derivatives of $\beta_1^e, \ldots, \beta_k^e$.

Thus, the natural form is

$$W_{\text{reg}} = W_0 + \sum_{e=1}^k \left( \frac{1}{\xi_e^{k+1}} \left( \xi_e \beta_e^e + B_e (\beta_e^e) \right) +$$

$$+ \frac{1}{2} \sum_{m, n, g=1}^L \frac{\partial \Omega_{mg} \beta_m^x \partial \beta_m^x}{\partial \beta_e^e} + \frac{1}{2} \sum_{m, n, p, q=1}^L \Omega_{mnpq} \frac{\partial \beta_m^x \partial \beta_n^x}{\partial \beta_e^e} +$$

$$+ \frac{1}{3} \sum_{m, n, p, q, r=1}^L \Omega_{mnpq} \frac{\partial \beta_m^x \partial \beta_n^x \partial \beta_p^x}{\partial \beta_e^e}$$

$$+ \sum_{e=k+1}^{2q} \frac{1}{\xi_e^{k+1}} \left( \xi_e \beta_e^e + B_e (\beta_e^e) \right) + C_e \left( \frac{\partial \beta_e^e}{\partial \xi_e} \right)^2 \right)$$
Euler–Lagrange equations
for $\beta^1, \ldots, \beta^k$

$$
\sum_{\nu, \rho, q = 1}^K \Theta_{\nu \rho \iota} \frac{\partial^2 \beta^\iota}{\partial x^\nu \partial x^\rho} - \sum_{\nu, \rho = 1}^K \Theta_{\nu \rho \iota \mu} \frac{\partial \beta^\mu}{\partial x^\rho} = \\
= \xi^{\iota} + (n+2) B_i \beta_i (\beta_i^{n+1})^{n+1}
$$

Simpler cases: $K = 2$, $\Theta_{\nu \rho \iota} = 0$

$$
\begin{align*}
\frac{\partial \beta^\iota}{\partial x^\iota} &= D_{121} \xi + C \eta + (n+2) B_2 \beta_2 (\beta_2^{n+1})^{n+1}, \\
\frac{\partial \beta^\iota}{\partial x^\rho} &= -D_{122} \xi - A \eta - (n+2) B_1 \beta_1 (\beta_1^{n+1})^{n+1}
\end{align*}
$$

where $\xi_1 = D_{121} \xi + A \eta$, $\xi_2 = D_{122} \xi + C \eta$

$A, C$ - arbitrary constants
For $n=2$, $A=C=0$

it is the special Riccati system

For $n=2$, $A=C=0$

- two-component extension of Abel equation

The case $K=2$, $D_{mpq}=0$

the system restricted to subspace $\xi_1=\xi_2=0$

is

$$\frac{\partial \beta_1^{\xi_c}}{\partial \xi} = A_{11} \beta_1^{\xi_1} + A_{12} \beta_2^{\xi_1} + A_{2} \xi,$$

$$\frac{\partial \beta_2^{\xi_c}}{\partial \xi} = A_{21} \beta_1^{\xi_1} + A_{22} \beta_2^{\xi_1} + A_{2} \xi,$$

where $\xi = \frac{1}{2} (\xi_1 + \xi_2)$ and $A_{ik}, A_{2} = \ldots$

At $n=2$. - two-component extension of Painlevé-I equation?
Possible extensions

Generic polynomial $P(z, \xi)$

$$P(z, \xi) = z^n + \sum_{k=1}^{N} z^{n-k} \xi_k.$$ 

General factorization

$$P(z, \xi) = \prod_{i=1}^{r} (z - x_i)^{m_i}.$$ 

where $\sum_{i=1}^{r} m_i = N.$

$M_m$ - set of all $P(z, \xi)$ of such form with fixed $\vec{m} = (m_1, \ldots, m_r).$

from

$$\prod_{i=1}^{r} (z - x_i)^{m_i} = z^n + \sum_{k=1}^{N} z^{n-k} \xi_k$$

$N-r$ constraints

$$\sum_{k} (\xi_1, \xi_2, \ldots, \xi_N) = 0, \quad k = 1, \ldots, N-r$$
Function \( W_{m,k,I}(x_i, z) \)

\[
W_{m,k,I} = \int_\Gamma \frac{dz}{2\pi i} \prod_{i \in I} (z - x_i)^{(1-k)m_i} (P(z, \bar{z}))^k.
\]

where

\( I \subset \{ 1, 2, \ldots, r \} \)

and \( k \) is real positive such that

\[ 0 < (1-k)m_i < 1, \quad i \in I \]

\( km_i \) - integer, \( i \notin I \)

**EPO system**

\[
(x_i - x_j) \frac{\partial^2 W_{m,k,I}}{\partial x_i \partial x_j} = (k-1)
\left( m_j \frac{\partial W_{m,k,I}}{\partial x_i} - m_i \frac{\partial W_{m,k,I}}{\partial x_j} \right)
\]

**Proposition.**

\[
\frac{\partial W_{m,k,I}}{\partial x_i} = 0 \iff x_i = x_i(\bar{z}).
\]